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ON COLOCATED CLUSTERED FINITE VOLUME SCHEMES FOR INCOMPRESSIBLE FLOW PROBLEMS

R. Eymard[1], R. Herbin[2], J.C. Latché[3] and B. Piar[4]

Abstract. We present and analyse in this paper a novel cell-centered colocated finite volume scheme for incompressible flows. Its definition involves a partition of the set of control volumes; each element of this partition is called a cluster and consists in a few neighbouring control volumes. Under a simple geometrical assumption for the clusters, we obtain that the pair of discrete spaces associating the classical cell-centered approximation for the velocities and cluster-wide constant pressures is *inf-sup* stable; in addition, we prove that a stabilization involving pressure jumps only across the internal edges of the clusters yields a stable scheme with the usual colocated discretization (*i.e.*, in particular, with control-volume-wide constant pressures), for the Stokes and the Navier-Stokes problem. An analysis of this stabilized scheme yields the existence of the discrete solution (and uniqueness for the Stokes problem). The convergence of the approximate solution to the solution of the continuous problem as the mesh size tends to zero is proven, provided, in particular, that the approximation of the mass balance flux is second order accurate; this condition imposes some geometrical conditions on the mesh. Under the same assumption, an error analysis is provided for the Stokes problem: it yields first-order estimates in energy norms. Numerical experiments confirm the theory and show, in addition, a second order convergence for the velocity in a discrete L^2 norm.

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1. Introduction

The use of Colocated Finite Volumes (CFV) for fluid flow problems is appealing for several reasons. Among them, let us mention a very cheap assembling step (in particular compared to finite elements, because there is no numerical integration to perform), the possibility to use, at least to some extent, general unstructured meshes with a low complexity of the data structure (compared with staggered schemes) suitable for the implementation of adaptative mesh refinement strategies and, finally, an easy coupling with additional conservation laws solvers, when these latters are developped within the finite volume framework. These features make CFV attractive for industrial problems, and they are widely used in Computational Fluid Dynamics, either in commercial

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(FLUENT, CFX, ...) or in proprietary codes, as encountered for instance in nuclear safety [1], which is a part of the context of this study.

When applied to incompressible flow problems, cell-centered colocated finite volumes suffer from a lack of coercivity, which was shown in [13,14] to be cured by a stabilization similar to the Brezzi-Pitkäranta technique, wellknown in the finite element context. When this stabilization is used, the existence of a solution to the discrete problem (unique in the linear case, *i.e.* the Stokes problem) is ensured, together with its convergence to the solution of the continuous problem, in both the steady and unsteady cases; for the steady Stokes problem and particular meshes, first order error estimates in natural (energy) norms are given in [14].

However, at high Reynolds numbers, numerical experiments show that the Brezzi-Pitkäranta stabilization term necessary to avoid pressure oscillations severely injures the accuracy of the solution. To overcome this problem, the Colocated Clustered Finite Volume scheme was introduced [5]. The idea of this scheme is to introduce a stabilization designed to prevent the short wavelengths oscillations of the pressure within a given cluster (*i.e.* a small group of control volumes), since the original equations are indeed sufficient to prevent from the long wavelength ones. In fact, one could even imagine to consider pressures which are constant on the clusters, but this turns out to be (numerically) not so favorable in terms of accuracy; moreover, the principle of one pressure per control volume is easier to implement, as the pressure then shares the discretization (and thus the same computer data structures) as other variables. Following these ideas, the cluster stabilization which was implemented in [5] consists in using a penalization of the pressure jumps only across the edges located within each cluster. This scheme gives very high quality results both for the Boussinesq approximation at high Reynolds numbers and the low Mach number approximation.

The goal of the present paper is to study the mathematical properties of this so–called "Colocated Clustered Finite Volume" scheme. Concerning the stability issue, our results are two-folds: first, we prove that a simple geometrical property for the clusters is equivalent to the *inf-sup* stability (e.g. [18]) of the pair of approximation spaces obtained by combining the standard cell-centered approximation for the velocity and an approximation of the pressure piecewise constant over each cluster; then this property is shown to yield the stability of the scheme. Under the same additional regularity property of the mesh as in [14], which seems in practice rather restrictive, we prove, with the analysis tools of [9], the convergence of the velocity and of the pressure towards the exact solution as the mesh size tends to 0, for both the steady Stokes and Navier-Stokes equations. In addition, we also obtain a first order error estimate in natural norms for the Stokes problem.

The paper is organized as follows: in section 2, we present the considered continuous problems and the weak formulations which are used in the subsequent analysis. In section 3, we define the discretization spaces and recall some fundamental results on the finite volume schemes. The Colocated Clustered Finite Volume scheme is then presented and analysed for the Stokes problem in section 4, and for the Navier-Stokes equations in section 5. Some numerical results are presented in section 6.

2. The continuous problem

We are interested in this paper in finding an approximation of the fields $\bar{u} = (\bar{u}^{(i)})_{i=1,\ldots,d} : \Omega \to \mathbb{R}^d$, and $\bar{p} : \Omega \to \mathbb{R}$, weak solution to the generalized incompressible steady Navier-Stokes equations which read:

$$\begin{aligned} &\eta\bar{u}^{(i)} + \sum_{j=1}^{d} \bar{u}^{(j)}\partial_j\bar{u}^{(i)} - \nu\Delta\bar{u}^{(i)} + \partial_i\bar{p} = f^{(i)} \text{ in } \Omega, \quad \text{for } i = 1,\ldots,d, \\ &\mathrm{div}\bar{u} = \sum_{i=1}^{d} \partial_i\bar{u}^{(i)} = 0 \text{ in } \Omega \times (0,T) \end{aligned} \tag{1}$$

with a homogeneous Dirichlet boundary condition for $\bar{u}$. In the above equations, $\bar{u}^{(i)}$, $i = 1,\ldots,d$ denote the components of the velocity of a fluid which flows in a domain Ω, ∂_i stands for the partial derivative with respect to the i^{th} variable, $\bar{p}$ denotes the pressure, $\nu > 0$ stands for the viscosity of the fluid. In tensorial form, the first equation of (1) equivalently reads:

$$\eta\bar{u} + (\bar{u}\cdot\nabla)\bar{u} - \nu\Delta\bar{u} + \nabla\bar{p} = f$$

We make the following assumptions:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^d,\ d = 2 \text{ or } 3 \tag{2}$$

$$\nu \in (0,+\infty),\ \eta \in [0,+\infty), \tag{3}$$

$$f \in \mathrm{L}^2(\Omega)^d. \tag{4}$$

We denote by $x = (x^{(i)})_{i=1,\ldots,d}$ any point of Ω, by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$, and by $\mathrm{d}x$ the d-dimensional Lebesgue measure $\mathrm{d}x = \mathrm{d}x^{(1)}\ldots\mathrm{d}x^{(d)}$.

The weak sense that we consider for the Navier-Stokes equations is the following.

Definition 2.1 (Weak solution to the steady Navier-Stokes equations)**.** Under hypotheses (2)-(4), let the function space $E(\Omega)$ be defined by:

$$E(\Omega) := \{\bar{v} = (\bar{v}^{(i)})_{i=1,\ldots,d} \in \mathrm{H}^1_0(\Omega)^d, \mathrm{div}\bar{v} = 0 \text{ a.e. in } \Omega\} \tag{5}$$

Then $\bar{u}$ is called a weak solution of (1) with a homogeneous Dirichlet boundary condition if $\bar{u} \in E(\Omega)$ and:

$$\begin{aligned}&\forall\varphi \in E(\Omega)\cap C_c^\infty(\Omega)^d,\\ &\qquad \eta\int_\Omega \bar{u}(x)\cdot\varphi(x)\,\mathrm{d}x + \nu\int_\Omega \nabla\bar{u}(x):\nabla\varphi(x)\,\mathrm{d}x\ + b(\bar{u},\bar{u},\varphi) = \int_\Omega f(x)\cdot\varphi(x)\,\mathrm{d}x\end{aligned} \tag{6}$$

where, for all $\bar{u},\bar{v} \in \mathrm{H}^1_0(\Omega)^d$ and for a.e. $x \in \Omega$, we use the following notation:

$$\nabla\bar{u}(x):\nabla\bar{v}(x) = \sum_{i=1}^d \nabla\bar{u}^{(i)}(x)\cdot\nabla\bar{v}^{(i)}(x)$$

and where the trilinear form $b(.,.,.)$ is defined, for all $\bar{u},\bar{v},\bar{w} \in (\mathrm{H}^1_0(\Omega))^d$, by:

$$b(\bar{u},\bar{v},\bar{w}) = \sum_{k=1}^d\sum_{i=1}^d \int_\Omega \bar{u}^{(i)}(x)\partial_i\bar{v}^{(k)}(x)\bar{w}^{(k)}(x)\,\mathrm{d}x = \int_\Omega (\bar{u}\cdot\nabla)\bar{v}\cdot\bar{w}\,\mathrm{d}x \tag{7}$$

We shall first analyze a scheme for the related linear problem, namely the generalized Stokes equations, which read:

$$\left|\begin{array}{l}\eta\bar{u} - \nu\Delta\bar{u} + \nabla\bar{p} = f \text{ in } \Omega\\[1ex] \mathrm{div}\bar{u} = g \text{ in } \Omega\end{array}\right. \tag{8}$$

where g is a source term supposed to belong to $\mathrm{L}^2(\Omega)$.

We then consider the following weak sense for this problem.

Definition 2.2 (Weak solution for the steady Stokes equations)**.** Under hypotheses (2)-(4), $(\bar{u},\bar{p})$ is called a weak solution of (8) if:

$$\left|\begin{array}{ll}\bar{u}\in \mathrm{H}_0^1(\Omega)^d,\ \bar{p}\in \mathrm{L}^2(\Omega) \text{ with } \displaystyle\int_\Omega \bar{p}(x)\mathrm{d}x=0 \text{ and:} & \\ \eta\displaystyle\int_\Omega \bar{u}(x)\cdot\bar{v}(x)\,\mathrm{d}x+\nu\int_\Omega \nabla\bar{u}(x):\nabla\bar{v}(x)\,\mathrm{d}x-\int_\Omega \bar{p}(x)\ \mathrm{div}\bar{v}(x)\,\mathrm{d}x=\int_\Omega f(x)\cdot\bar{v}(x)\,\mathrm{d}x & \forall\bar{v}\in \mathrm{H}_0^1(\Omega)^d \\ \displaystyle\int_\Omega \bar{q}(x)\ \mathrm{div}\bar{u}(x)\mathrm{d}x=\int_\Omega g(x)\,\bar{q}(x)\ \mathrm{d}x & \forall\bar{q}\in \mathrm{L}^2(\Omega)\end{array}\right. \tag{9}$$

The existence and uniqueness of the weak solution of (8) in the sense of the above definition is a classical result (again, see e.g. [23] or [3]).

3. Spatial discretization and discrete functional analysis

3.1. **Admissible discretization of the computational domain**

We first enrich the definition of admissible discretization for a finite volume method given in [9] by introducing the notion of cluster. The first three items of the following definition are thus classical, and only the last one is new.

Definition 3.1 (Admissible discretization)**.** Let Ω be an open bounded polygonal (polyhedral if $d=3$) subset of $\mathbb{R}^d$, and $\partial\Omega=\overline{\Omega}\setminus\Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D}=(\mathcal{M},\mathcal{E},\mathcal{P},\mathcal{G})$, where:

- $\mathcal{M}$ is a finite family of non empty open polygonal convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega}=\cup_{K\in\mathcal{M}}\overline{K}$. For any $K\in\mathcal{M}$, let $\partial K=\overline{K}\setminus K$ be the boundary of K and $m_K>0$ denote the area of K.
- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma\in\mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^d$ and $K\in\mathcal{M}$ with $\overline{\sigma}=\partial K\cap E$ and σ is a non empty open subset of E. We then denote by $m_\sigma>0$ the (d-1)-dimensional measure of σ. We assume that, for all $K\in\mathcal{M}$, there exists a subset $\mathcal{E}(K)$ of $\mathcal{E}$ such that $\partial K=\cup_{\sigma\in\mathcal{E}(K)}\overline{\sigma}$. It then results from the previous hypotheses that, for all $\sigma\in\mathcal{E}$, either $\sigma\subset\partial\Omega$ or there exists $(K,L)\in\mathcal{M}^2$ with $K\neq L$ such that $\overline{K}\cap\overline{L}=\overline{\sigma}$; we denote in the latter case $\sigma=K|L$.
- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P}=(x_K)_{K\in\mathcal{M}}$. The coordinates of x_K are denoted by $x_K^{(i)}$, $i=1,\dots,d$. The family $\mathcal{P}$ is such that, for all $K\in\mathcal{M}$, $x_K\in K$. Furthermore, for all $\sigma\in\mathcal{E}$ such that there exists $(K,L)\in\mathcal{M}^2$ with $\sigma=K|L$, it is assumed that the straight line (x_K,x_L) going through x_K and x_L is orthogonal to $K|L$.
- $\mathcal{G}$ is a partition of $\mathcal{M}$ (the elements of $\mathcal{G}$ are disjoint subsets of $\mathcal{M}$, the union of which is equal to $\mathcal{M}$).

In addition, we shall say that the mesh is "super-admissible", if, as in [14], for any internal $\sigma=K|L$, the line (x_K,x_L) meets $K|L$ at its center of gravity.

An example of two cells of an admissible mesh is given in Figure 1, along with some of the notations which we now introduce.

For all $G\in\mathcal{G}$, we note $C_G=\bigcup_{K\in G}K$. We define two parameters to characterize the size of the discretization, $h_\mathcal{M}$ and $h_\mathcal{G}$, as, respectively the maximal diameter of the control volumes and clusters:

$$h_\mathcal{M}=\sup_{K\in\mathcal{M}}h_K \qquad\qquad h_\mathcal{G}=\sup_{G\in\mathcal{G}}h_{C_G}$$

where h_K and h_{C_G} stands respectively for the diameter of the control volume K and of the cluster C_G.

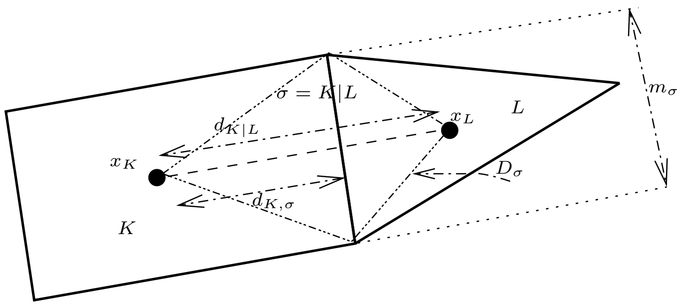


FIGURE 1. Notations for two neighbouring control volumes

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, we denote by $n_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$), that is $\mathcal{E}_{\rm int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{M}$, we denote by $\mathcal{N}_K$ the subset of $\mathcal{M}$ of the neighbouring control volumes to K. For all $K \in \mathcal{M}$ and $L \in \mathcal{N}_K$, we set $n_{K|L} = n_{K,K|L}$, we denote by $d_{K|L}$ the Euclidean distance between x_K and x_L; the notation is extended to any external edge σ of a control volume K, for which we set $d_\sigma = d_{K,\sigma}$. For each edge σ of any control volume K, we denote by $D_{K,\sigma}$ the volume defined by:

$$D_{K,\sigma} = \{t\,x + (1-t)\,x_K,\ x \in \sigma,\ t \in (0,1)\}$$

The so–called diamond-cell associated the edge σ is defined by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ when $\sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L$ and $D_\sigma = D_{K,\sigma}$ when $\sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K)$.

For all $K \in \mathcal{M}$, we denote by G_K the unique element of $\mathcal{G}$ such that $K \in G_K$. Let us note that we cannot have: $\forall K \in \mathcal{M}$, $\mathcal{N}_K \subset G_K$, since $(G_K)_{K\in\mathcal{M}}$ is a partition of $\mathcal{M}$ while $(\mathcal{N}_K)_{K\in\mathcal{M}}$ is not. But indeed, for any $K \in \mathcal{M}$, on has $\mathcal{N}_K \cap G_K \neq \emptyset$, and we shall assume that the clusters satisfy the following geometrical property:

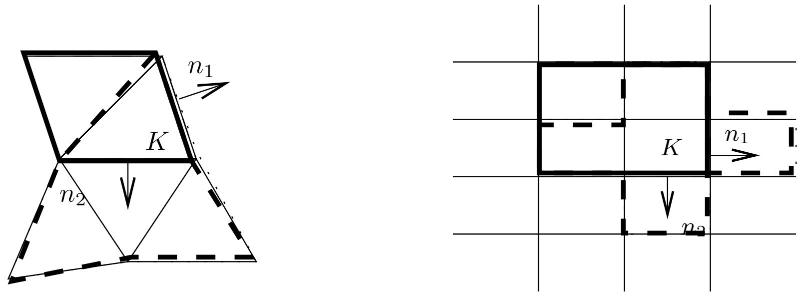
$$\forall K \in \mathcal{M} \text{ such that } \mathcal{N}_K \not\subset G_K, \qquad \sum_{L\in\mathcal{N}_K\setminus G_K} a_L\ n_{K|L} = 0 \quad \Rightarrow \quad \forall L \in \mathcal{N}_K \setminus G_K,\ a_L = 0 \tag{10}$$

In figure 2 below, we show examples of clusters which satisfy (top line of the picture) this property (for triangles and rectangles), and which do not satisfy this property (bottom line). Roughly speaking, assumption (10) implies that there should not be too many edges of a cell which are "outside" of the cluster to which this cell belongs. We will prove hereafter that this very simple geometrical relation is equivalent to the *inf-sup* stability of the pair of approximation spaces composed of the classical cell-centered finite volume approximation for the velocity and an approximation of the pressure piecewise constant over each cluster. We then set:

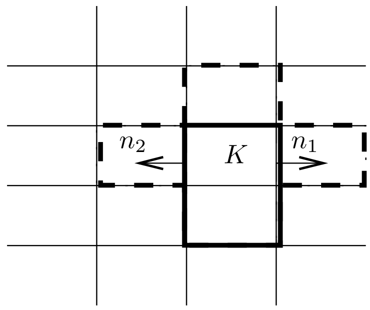
$$\begin{aligned} &\mathrm{regul}(\mathcal{D},\mathcal{G}) = \inf_{K\in\mathcal{M},\ \mathcal{N}_K\not\subset G_K} I_K \quad \text{ with} \\ &I_K = \inf\{\ [\sum_{L\in\mathcal{N}_K\setminus G_K} a_L n_{K|L}]^2,\ \forall\ (a_L)_{L\in\mathcal{N}_K\setminus G_K} \subset \mathbb{R} \text{ such that } \sum_{L\in\mathcal{N}_K\setminus G_K} a_L^2 = 1\}. \end{aligned} \tag{11}$$

We shall measure the regularity of the mesh through the function $\mathrm{regul}(\mathcal{D})$ defined by:

$$\begin{aligned} \mathrm{regul}(\mathcal{D}) = \inf\ &\{\mathrm{regul}(\mathcal{D},\mathcal{G})\}\ \cup\ \{\frac{m_\sigma}{h_K^{d-1}}, \frac{d_{K,\sigma}}{h_K},\ K \in \mathcal{M},\ \sigma \in \mathcal{E}(K)\} \\ &\cup \{\frac{d_{K,K|L}}{d_{K|L}},\ K \in \mathcal{M},\ L \in \mathcal{N}_K\}\ \cup\ \{\frac{h_K}{h_L},\ K \in \mathcal{M},\ L \in \mathcal{N}_K\}\ \cup\ \{\frac{1}{\mathrm{card}(\mathcal{E}(K))},\ K \in \mathcal{M}\} \end{aligned} \tag{12}$$



Admissible clusters: $a_1 n_1 + a_2 n_2 = 0 \Rightarrow a_1 = a_2 = 0$



Non admissible cluster: $n_1 + n_2 = 0$

FIGURE 2. Admissible and non admissible clusters. For a given cell K, the set of cells defining the cluster G_K is outlined with a solid bold line, and the set $\mathcal{N}_K$ of neighbouring cells to K is outlined with a dashed bold line

Super–admissible discretizations $\mathcal{D}$ such that $\mathrm{regul}(\mathcal{D}) > 0$ are easily encountered: for example, one can consider in 2D (resp. 3D) a rectangular (resp. parallelepipedic) mesh, each cluster being defined by the rectangles (resp. parallelepipeds) sharing a same vertex. Another example is a mesh of triangles with all angles acute, in order that the circumcenter of each triangle be located inside the triangle, and the clusters are pairs of triangles. Both examples are depicted in the first line of Figure 2. In 3D however, the assumption of "super–admissibility" is unfortunately known to be satisfied only by parallelelipedic meshes. Note that the scheme is shown to be stable without this assumption; indeed the super–admissibility is only required in our convergence proof in order to obtain enough consistency on the divergence operator (estimate (42) of theorem 4.7).

3.2. **Discrete functional properties**

The space $\mathrm{H}_{\mathcal{D}}(\Omega)$

Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be an admissible finite volume discretization of Ω in the sense of definition 3.1. We denote by $\mathrm{H}_{\mathcal{D}}(\Omega) \subset \mathrm{L}^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $w \in \mathrm{H}_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by w_K the constant value of w in K.

Bilinear forms, norms and semi-norms associated to $\mathrm{H}_{\mathcal{D}}(\Omega)$

For $(v,w)\in(\mathrm{H}_{\mathcal{D}}(\Omega))^2$, we define the following inner product, which is the discrete analogue of the canonical $\mathrm{H}^1_0(\Omega)$ bilinear form:

$$[v,w]_{\mathcal{D}}=\sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\ \sigma=K|L}\frac{m_\sigma}{d_\sigma}(v_L-v_K)(w_L-w_K)+\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}},\ \sigma\in\mathcal{E}(K)}\frac{m_\sigma}{d_{K,\sigma}}v_K w_K \tag{13}$$

We then obtain a norm in $\mathrm{H}_{\mathcal{D}}(\Omega)$ (thanks to the discrete Poincaré inequality (14) given below) by:

$$\|w\|_{\mathcal{D}}=([w,w]_{\mathcal{D}})^{1/2}$$

These definitions naturally extend to vector valued functions as follows. For $u=(u^{(i)})_{i=1,\dots,d}\in(\mathrm{H}_{\mathcal{D}}(\Omega))^d$, $v=(v^{(i)})_{i=1,\dots,d}\in(\mathrm{H}_{\mathcal{D}}(\Omega))^d$ and $w=(w^{(i)})_{i=1,\dots,d}\in(\mathrm{H}_{\mathcal{D}}(\Omega))^d$, we define:

$$\|u\|_{\mathcal{D}}=\left(\sum_{i=1}^{d}[u^{(i)},u^{(i)}]_{\mathcal{D}}\right)^{1/2},\qquad [v,w]_{\mathcal{D}}=\sum_{i=1}^{d}[v^{(i)},w^{(i)}]_{\mathcal{D}}$$

The discrete Poincaré inequality (see [9]) reads:

$$\|w\|_{\mathrm{L}^2(\Omega)}\le \mathrm{diam}(\Omega)\ \|w\|_{\mathcal{D}},\ \forall w\in\mathrm{H}_{\mathcal{D}}(\Omega) \tag{14}$$

We define a discrete $\mathrm{H}^{-1}(\Omega)^d$ norm, which reads, for any function f of $\mathrm{L}^2(\Omega)^d$:

$$\|f\|_{-1,\mathcal{D}}=\sup_{v\,\in\,\mathrm{H}_{\mathcal{D}}(\Omega)^d}\frac{\displaystyle\int_\Omega f(x)\cdot v(x)\ \mathrm{d}x}{\|v\|_{\mathcal{D}}}$$

By the discrete Poincaré inequality, we obtain that $\|f\|_{-1,\mathcal{D}}\le \mathrm{diam}(\Omega)\ \|f\|_{\mathrm{L}^2(\Omega)^d}$.

Finally, we define the three following bilinear forms over $\mathrm{H}_{\mathcal{D}}(\Omega)\times\mathrm{H}_{\mathcal{D}}(\Omega)$ by the following relations:

$$\begin{aligned}
\langle v,w\rangle_{\mathcal{M}} &=\frac{1}{2}\sum_{K\in\mathcal{M}}\ \sum_{L\in\mathcal{N}_K} m_{K|L}(h_K+h_L)(v_L-v_K)(w_L-w_K)\\
\langle v,w\rangle_{\mathcal{G}} &=\frac{1}{2}\sum_{K\in\mathcal{M}}\ \sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}(h_K+h_L)(v_L-v_K)(w_L-w_K)\\
\langle v,w\rangle_{\mathcal{M}\setminus\mathcal{G}} &=\frac{1}{2}\sum_{K\in\mathcal{M}}\ \sum_{L\in\mathcal{N}_K\setminus G_K} m_{K|L}(h_K+h_L)(v_L-v_K)(w_L-w_K)
\end{aligned} \tag{15}$$

Note that $\langle v,w\rangle_{\mathcal{M}}=\langle v,w\rangle_{\mathcal{G}}+\langle v,w\rangle_{\mathcal{M}\setminus\mathcal{G}}$. Each of these three bilinear forms defines a semi-norm over $\mathrm{H}_{\mathcal{D}}(\Omega)$:

$$|w|_{\mathcal{M}}=\langle w,w\rangle_{\mathcal{M}}^{1/2}\qquad |w|_{\mathcal{G}}=\langle w,w\rangle_{\mathcal{G}}^{1/2}\qquad |w|_{\mathcal{M}\setminus\mathcal{G}}=\langle w,w\rangle_{\mathcal{M}\setminus\mathcal{G}}^{1/2} \tag{16}$$

Interpolation operators

We define the interpolation operator $P_{\mathcal{D}}$, mapping $C(\Omega)$ onto $\mathrm{H}_{\mathcal{D}}(\Omega)$, by setting $(P_{\mathcal{D}}\varphi)_K=\varphi(x_K)$, for all $K\in\mathcal{M}$, for all $\varphi\in C(\Omega)$. Its natural extension to vector valued functions, also noted $P_{\mathcal{D}}$, maps $C(\Omega)^d$ onto $\mathrm{H}_{\mathcal{D}}(\Omega)^d$, by $(P_{\mathcal{D}}\varphi)_K=\varphi(x_K)$, for all $K\in\mathcal{M}$, for all $\varphi\in C(\Omega)^d$.

For possibly discontinuous functions, we define two additional interpolation operators, $P_{\mathcal{M}}$ and $P_{\mathcal{G}}$. The first one maps $\mathrm{L}^2(\Omega)$ onto $\mathrm{H}_{\mathcal{D}}(\Omega)$, the second one maps $\mathrm{L}^2(\Omega)$ onto the sub-space of functions of $\mathrm{H}_{\mathcal{D}}(\Omega)$ which are constant over each cluster:

$$(P_{\mathcal{M}}\varphi)_K = \frac{1}{m_K}\int_K \varphi(x)\;\mathrm{d}x \qquad\qquad (P_{\mathcal{G}}\varphi)_K = \frac{1}{\mathrm{meas}(C_{G_K})}\int_{C_{G_K}} \varphi(x)\;\mathrm{d}x \tag{17}$$

where $\mathrm{meas}(C_{G_K})$ stands fo the measure of the cluster C_{G_K}.

The operator $P_{\mathcal{M}}$ satisfies the following continuity result, the proof of which easily follows from estimates (73) and (74).

Lemma 3.2. *Let assumption* (2) *hold, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Let $v \in \mathrm{H}^1_0(\Omega)^d$. Then the following bound holds:*

$$\|P_{\mathcal{M}}(v)\|_{\mathcal{D}} \le c\;|v|_{\mathrm{H}^1(\Omega)^d}$$

where c only depends on Ω and θ and $|\cdot|_{\mathrm{H}^1(\Omega)^d}$ stands for the H^1 *seminorm on* Ω.

4. Approximation of the generalized Stokes problem

4.1. The finite volume scheme

Finite volume schemes are classically presented as discrete balance equations with a suitable approximation of the fluxes, see e.g. [9]. However, in recent works dealing with cell centered finite volume methods for elliptic problems [11], an equivalent variational formulation in adequate functional spaces is introduced, and this presentation is probably more convenient for the analysis of the schemes, as it is a natural starting point to derive stability estimates. We follow here this latter path.

We begin by defining a discrete divergence operator $\mathrm{div}_{\mathcal{D}}$, the expression of which is the same as in [14], and which maps $\mathrm{H}_{\mathcal{D}}(\Omega))^d$ onto $\mathrm{H}_{\mathcal{D}}(\Omega)$ and reads:

$$\mathrm{div}_{\mathcal{D}}(u)(x) = \frac{1}{m_K}\sum_{L\in\mathcal{N}_K} m_{K|L}\;n_{K|L}\cdot\frac{d_{L,\sigma}u_K + d_{K,\sigma}u_L}{d_{K|L}}, \qquad \text{for a.e. } x\in K,\ \forall K\in\mathcal{M} \tag{18}$$

The discrete analogue of the space of divergence-free vector fields consequently reads:

$$E_{\mathcal{D}}(\Omega) = \{u \in (\mathrm{H}_{\mathcal{D}}(\Omega))^d, \mathrm{div}_{\mathcal{D}}(u) = 0\}$$

The adjoint of this discrete divergence defines a discrete gradient $\nabla_{\mathcal{D}}$, mapping $\mathrm{H}_{\mathcal{D}}(\Omega)$ onto $(\mathrm{H}_{\mathcal{D}}(\Omega))^d$, which takes the expression:

$$(\nabla_{\mathcal{D}}p)_K(x) = \frac{1}{m_K}\sum_{L\in\mathcal{N}_K} m_{K|L}\;n_{K|L}\frac{d_{L,\sigma}}{d_{K|L}}(p_L - p_K), \qquad \text{for a.e. } x\in K,\ \forall K\in\mathcal{M} \tag{19}$$

As $\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma\;n_{K,\sigma} = 0$, this discrete gradient equivalently reads:

$$\begin{aligned}&(\nabla_{\mathcal{D}}p)_K = \frac{1}{m_K}\left[\sum_{L\in\mathcal{N}_K} F_{\mathrm{grad},K|L} + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} F_{\mathrm{grad},\sigma}\right] \qquad \text{with}\\ &F_{\mathrm{grad},K|L} = m_{K|L}\frac{d_{L,\sigma}p_L + d_{K,\sigma}p_K}{d_{K|L}}\;n_{K|L}, \qquad F_{\mathrm{grad},\sigma} = m_\sigma\,p_K\,n_{K,\sigma}\end{aligned} \tag{20}$$

in which we recognize a classical "flux-based" finite volume formulation, with, however, a rather unnatural (and only first order consistent) interpolation of the pressure on the edge.

The discrete solution is then defined as the pair of functions (u, p) such that:

$$\left| \begin{array}{ll} (u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\ \mathrm{d}x = 0 & \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\ \mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(v)(x)\ \mathrm{d}x = \int_\Omega f(x)\cdot v(x)\ \mathrm{d}x & \forall v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)\ q(x)\ \mathrm{d}x + \langle p,q\rangle_{\alpha,\lambda} = \int_\Omega g(x)\ q(x)\ \mathrm{d}x & \forall q \in \mathrm{H}_{\mathcal{D}}(\Omega) \end{array} \right. \tag{21}$$

where the bilinear form $\langle\cdot,\cdot\rangle_{\alpha,\lambda}$ corresponds to a "cluster-wide" stabilization, defined as follows:

$$\langle p,q\rangle_{\alpha,\lambda} = \frac{\lambda}{2} \sum_{K\in\mathcal{M}} \quad \sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}\ (h_K+h_L)^\alpha (v_L - v_K)(w_L - w_K) \tag{22}$$

λ and α being two parameters, respectively positive ($\lambda > 0$) and lower than one $\alpha \leq 1$. Note that $\langle p,q\rangle_{1,\lambda} = \lambda\ \langle p,q\rangle_{\mathcal{G}}$. The bilinear form $\langle\cdot,\cdot\rangle_{\alpha,\lambda}$ is associated to the following semi-norm:

$$|p|_{\alpha,\lambda} = \langle p,p\rangle_{\alpha,\lambda}^{1/2}$$

which satisfies the following inequality, provided that the diameter of each control volume is smaller than 1/2:

$$\langle p,q\rangle_{\alpha,\lambda} \geq \lambda\ \langle p,q\rangle_{\mathcal{G}}, \qquad \forall \alpha \leq 1 \tag{23}$$

We will see in the following that $\alpha = 1$ corresponds to the maximal value for α beyond which the stability of the scheme can no more be ensured.

System (21) is equivalent to searching for the family of vectors $(u_K)_{K\in\mathcal{M}}$ of $\mathbb{R}^d$, and scalars $(p_K)_{K\in\mathcal{M}}$ solution of the system of equations (written under flux form) obtained by writing for each control volume K of $\mathcal{M}$:

$$\left| \begin{array}{l} \eta\ m_K\ u_K - \nu \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\cap\mathcal{E}(K),\\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma}(u_L - u_K) - \nu \sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)} \frac{m_\sigma}{d_{K,\sigma}}(0 - u_K) \\ \qquad + \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma\ \frac{d_{L,\sigma}p_L + d_{K,\sigma}p_K}{d_\sigma}\ n_\sigma + \sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)} m_\sigma\ p_K\ n_{K,\sigma} = \int_K f(x)\ \mathrm{d}x \\ \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma\ \frac{d_{L,\sigma}u_K + d_{K,\sigma}u_L}{d_\sigma}\cdot n_\sigma - \lambda \sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}(h_K+h_L)^\alpha (p_L - p_K) = \int_K g(x)\ \mathrm{d}x \end{array} \right. \tag{24}$$

supplemented by the relation:

$$\sum_{K\in\mathcal{M}} m_K\ p_K = 0 \tag{25}$$

4.2. **Stability of the scheme**

This section is aimed at proving the stability of the scheme; it is worth stressing that, to this purpose, the mesh is not required to be "super–admissible", but only admissible in the sense of definition 3.1.

We first begin by a (partial) stability result for the discrete gradient operator, which is not specific to a clustered approximation, and may be seen as the "finite volume analogue" to a lemma alread known in the finite element context [16, 24].

Lemma 4.1. *Under hypotheses* (2)-(4)*, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1. Let $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Then there exists two strictly positive real numbers β_1 and β_2 depending only on d, Ω and θ such that the following holds:*

$$\forall p \in \mathrm{H}_{\mathcal{D}}(\Omega) \text{ with } \int_\Omega p(x)\ \mathrm{d}x = 0,\ \exists v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$$
$$\text{such that} \quad \left|\ \begin{array}{l} \|v\|_{\mathcal{D}} \leq \beta_1 \|p\|_{\mathrm{L}^2(\Omega)} \\ \displaystyle\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}} v(x)\ \mathrm{d}x \geq \|p\|^2_{\mathrm{L}^2(\Omega)} - \beta_2\ |p|^2_{\mathcal{M}} \end{array}\right. \tag{26}$$

Proof. Let $p \in \mathrm{H}_{\mathcal{D}}(\Omega)$ be given. We apply a result by Nečas [20]. Because $p(x)$ is a zero mean-valued function, there exists c_{dr}, which only depends on d and Ω, and $\bar v \in \mathrm{H}^1_0(\Omega)^d$ such that $\mathrm{div}\bar v(x) = p(x)$ for a.e. $x \in \Omega$ and:

$$\|\bar v\|_{\mathrm{H}^1(\Omega)^d} \leq c_{\mathrm{dr}}\ \|p\|_{\mathrm{L}^2(\Omega)} \tag{27}$$

We then set:

$$v_\sigma^{(i)} = \frac{1}{m_\sigma} \int_\sigma \bar v^{(i)}(x)\ \mathrm{d}\gamma(x), \qquad \forall \sigma \in \mathcal{E},\ \forall i = 1, \ldots, d$$

(note that $v_\sigma^{(i)} = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$ and $i = 1, \ldots, d$) and we define $v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ by:

$$v_K^{(i)} = \frac{1}{m_K} \int_K \bar v^{(i)}(x)\ \mathrm{d}x, \qquad \forall K \in \mathcal{M},\ \forall i = 1, \ldots, d$$

As proved in the appendix (relation (73)), we know that there exists c_1 depending on d, Ω and θ such that:

$$(v_K - v_\sigma)^2 \leq c_1\ \frac{h_K}{m_\sigma} \int_K (\nabla \bar v(x))^2\ \mathrm{d}x \tag{28}$$

In addition, by the continuity of the interpolation operator $P_{\mathcal{M}}$ (lemma 3.2), there exists another real c_2 once again depending on d, Ω and θ such that:

$$\|v\|_{\mathcal{D}} \leq c_2\ \|\bar v\|_{\mathrm{H}^1(\Omega)^d} \leq c_2\ c_{\mathrm{dr}}\ \|p\|_{\mathrm{L}^2(\Omega)} \tag{29}$$

We then have:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}} v(x)\ \mathrm{d}x = \sum_{K \in \mathcal{M}} p_K \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \cap \mathcal{E}(K), \\ \sigma = K|L}} m_\sigma\ n_\sigma \cdot \frac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_\sigma} = T_1 + T_2$$

where:

$$\begin{aligned} T_1 &= \sum_{K \in \mathcal{M}} p_K \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \cap \mathcal{E}(K), \\ \sigma = K|L}} m_\sigma\ n_\sigma \cdot v_\sigma = \sum_{K \in \mathcal{M}} p_K \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \cap \mathcal{E}(K), \\ \sigma = K|L}} \int_\sigma \bar v(x) \cdot n_\sigma\ \mathrm{d}\gamma(x) \\ &= \int_\Omega p(x)\ \mathrm{div}\bar v(x)\ \mathrm{d}x = \|p\|^2_{\mathrm{L}^2(\Omega)} \end{aligned}$$

and:

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{M}} p_K \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma \left(\frac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_\sigma} - v_\sigma\right)\cdot n_\sigma \\ &= \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma (p_K - p_L) \left(\frac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_\sigma} - v_\sigma\right)\cdot n_\sigma \end{aligned}$$

We then have, thanks to the Cauchy-Schwarz inequality:

$$T_2^2 \leq \left[\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma\ (h_K+h_L)\ (p_K-p_L)^2\right] \left[\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} \frac{m_\sigma}{h_K+h_L} \left(\frac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_\sigma} - v_\sigma\right)^2\right]$$

Hence, remarking that, as, for any internal edge $\sigma = K|L$, $d_{K,\sigma} < d_\sigma$, $d_{L,\sigma} < d_\sigma$ and $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$:

$$(\frac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_\sigma} - v_\sigma)^2 \leq 2\ (v_K - v_\sigma)^2 + 2\ (v_L - v_\sigma)^2$$

we get by inequality (28) and (27), as the maximal number of edges for a control volume is bounded by assumption on θ:

$$T_2^2 \leq c_3\ |p|_{\mathcal{M}}\ \|\bar v\|_{\mathrm{H}^1(\Omega)^d} \leq c_3\ c_{\rm dr}\ |p|_{\mathcal{M}}\ \|p\|_{\mathrm{L}^2(\Omega)}$$

where c_3 only depends on d, Ω and θ. Collecting terms, we obtain by Young's inequality:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}} v(x)\ \mathrm{d}x \geq \frac{1}{2}\ \|p\|^2_{\mathrm{L}^2(\Omega)} - \frac{(c_3\ c_{\rm dr})^2}{2} |p|^2_{\mathcal{M}}$$

which, together with relation (29), concludes the proof. □

The following result is an essential stability feature of the scheme. It proves, in particular, that the simple geometrical regularity of the mesh enforced by relation (11), is equivalent to the inf-sup stability of the cluster-wide constant pressure spaces (see remark below).

Lemma 4.2. *Under hypotheses* (2)-(4)*, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1. Let $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Then there exist two positive real numbers, again denoted by β_1 and β_2, depending only on d, Ω and θ such that the following holds:*

$$\forall p \in \mathrm{H}_{\mathcal{D}}(\Omega)\ \textit{with} \int_\Omega p(x)\ \mathrm{d}x = 0,\ \exists v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \quad \textit{such that} \left|\begin{array}{l} \|v\|_{\mathcal{D}} \leq \beta_1 \|p\|_{\mathrm{L}^2(\Omega)} \\ \displaystyle\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}} v(x)\ \mathrm{d}x \geq \|p\|^2_{\mathrm{L}^2(\Omega)} - \beta_2\ |p|^2_{\mathcal{G}} \end{array}\right. \tag{30}$$

Proof. Let $p \in \mathrm{H}_{\mathcal{D}}(\Omega)$ be given. We define $v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ by:

$$v_K = \frac{1}{h_K^{d-2}} \sum_{L\in\mathcal{N}_K\setminus G_K} m_{K|L}\ \frac{d_{L,K|L}}{d_{K|L}} (p_L - p_K)\ n_{K|L}$$

As the discrete divergence is the transposed of the discrete gradient, we have:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(-v)(x)\ \mathrm{d}x = \int_\Omega \nabla_{\mathcal{D}} p(x) \cdot v(x)\ \mathrm{d}x =$$
$$\sum_{K\in\mathcal{M}} v_K \cdot [\underbrace{\sum_{L\in\mathcal{N}_K\setminus G_K} m_{K|L}\ \frac{d_{L,K|L}}{d_{K|L}}(p_L - p_K)\ n_{K|L}}_{T_{1,K}} + \underbrace{\sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}\ \frac{d_{L,K|L}}{d_{K|L}}(p_L - p_K)\ n_{K|L}}_{T_{2,K}}\ ]$$

Remarking that $v_K = \dfrac{1}{h_K^{d-2}}\ T_{1,K}$, we have by Young's inequality:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(-v)(x)\ \mathrm{d}x \geq \frac{1}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}}\ (T_{1,K}^2 - T_{2,K}^2)$$

Thanks to the regularity assumption on the mesh (in particular definition (11)), the first part of this summation satisfies the following estimate:

$$\begin{aligned}
\frac{1}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}}\ T_{1,K}^2 \quad &= \frac{1}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}} \left[\sum_{L\in\mathcal{N}_K\setminus G_K} m_{K|L}\ \frac{d_{L,K|L}}{d_{K|L}}(p_L - p_K)\ n_{K|L} \right]^2 \\
&\geq \frac{\theta}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}} \sum_{L\in\mathcal{N}_K\setminus G_K} m_{K|L}^2 \left(\frac{d_{L,K|L}}{d_{K|L}} \right)^2\ (p_L - p_K)^2 \\
&= \frac{\theta}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K\setminus G_K} \frac{m_{K|L}}{h_K^{d-2}\ (h_K + h_L)} \left(\frac{d_{L,K|L}}{d_{K|L}} \right)^2\ m_{K|L}\ (h_K + h_L)\ (p_L - p_K)^2
\end{aligned}$$

and thus, by regularity assumptions on the mesh, there exists $c_1(\theta) > 0$ depending only on θ such that:

$$\frac{1}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}}\ T_{1,K}^2 \geq c_1(\theta)\ |p|^2_{\mathcal{M}\setminus\mathcal{G}}$$

By a similar computation, we get:

$$\begin{aligned}
\frac{1}{2} \sum_{K\in\mathcal{M}} \frac{1}{h_K^{d-2}}\ T_{2,K}^2 \quad &\leq \frac{1}{2} \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{N}_K\cap G_K} \frac{m_{K|L}}{h_K^{d-2}\ (h_K + h_L)} \left(\frac{d_{L,K|L}}{d_{K|L}} \right)^2\ m_{K|L}\ (h_K + h_L)\ (p_L - p_K)^2 \\
&\leq c_2(\theta)\ |p|^2_{\mathcal{G}}
\end{aligned}$$

where, once again, $c_2(\theta)$ only depends on the regularity of the mesh. Thus, collecting the bounds, we get:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(-v)(x)\ \mathrm{d}x \geq c_1(\theta)\ |p|^2_{\mathcal{M}\setminus\mathcal{G}} - c_2(\theta)\ |p|^2_{\mathcal{G}} \tag{31}$$

On the other hand, because, by assumption, the number of edges of the control volumes is bounded, we have:

$$\begin{aligned}
\|v\|_{\mathcal{D}}^2 &\leq 2 \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} \frac{m_\sigma}{d_\sigma}\ (v_K^2+v_L^2) + \sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)} \frac{m_\sigma}{d_\sigma}\ v_K^2 \\
&\leq c \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} \frac{m_\sigma}{d_\sigma} \left[\sum_{M\in\mathcal{N}_K} \frac{(m_{K|M})^2}{h_K^{2\,(d-2)}}\ (p_K^2+p_M^2) + \sum_{M\in\mathcal{N}_L} \frac{(m_{L|M})^2}{h_K^{2\,(d-2)}}\ (p_M^2+p_L^2) \right] \\
&+ c \sum_{\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}(K)} \frac{m_\sigma}{d_\sigma} \sum_{M\in\mathcal{N}_K} \frac{m_\sigma^2}{h_K^{2\,(d-2)}}\ (p_K^2+p_M^2)
\end{aligned}$$

Reordering the summations, the above relation takes the following form:

$$\|v\|_{\mathcal{D}}^2 \leq \sum_{K\in\mathcal{M}} \mu_K\ p_K^2$$

and the regularity of the mesh implies that each coefficient $\mu_K \leq c\ m_K$ where c only depends on θ and d; note that we use here the assumption that the ratio of the measure of two neighbouring volumes is bounded. We thus get:

$$\|v\|_{\mathcal{D}}^2 \leq c_3(\theta)\ \|p\|_{\mathrm{L}^2(\Omega)} \tag{32}$$

Estimates (31) and (32) prove the existence of $v_1 \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ such that:

$$\int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(v_1)(x)\ \mathrm{d}x \geq \beta_2\ |p|_{\mathcal{M}\setminus\mathcal{G}}^2 - c_4(\theta)\ |p|_{\mathcal{G}}^2, \qquad \|v_1\|_{\mathcal{D}}^2 \leq c_5(\theta)\ \|p\|_{\mathrm{L}^2(\Omega)}$$

Let $v_2 \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ be such that the estimate of lemma 4.1 holds; then the desired result is obtained with $v = v_1 + v_2$. □

Remark 4.3. We easily note that $|p|_{\mathcal{G}}$ vanishes for pressure fields which are constant over each cluster. The previous result thus shows that combining an approximation of the velocity by the space $\mathrm{H}_{\mathcal{D}}(\Omega)^d$ and an approximation of the pressure by the functions in $\mathrm{H}_{\mathcal{D}}(\Omega)$ which are constant on each cluster yields an inf-sup stable discretization, which should be quite usable in practice. However, letting the pressure vary within the clusters and adding a stabilization term is both easier to implement and, from numerical experiments, more accurate.
This result suggests that the scheme under consideration may be in some particular cases obtained via a minimal stabilization procedure as defined in [4]; an example of such a derivation is given in [15].

We are now in position to prove stability estimates for the velocity and the pressure.

Theorem 4.4 (Estimates on the velocity and the pressure)**.** *We suppose that hypotheses* (2)-(4) *hold. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and let $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$. *Let $\lambda \in (0,+\infty)$ and $\alpha \leq 1$ be given. Let $(u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega)$ be a solution to* (21)*. Then, for any f_1 and f_2 in $\mathrm{L}^2(\Omega)^d$ such that $f = f_1 + f_2$, there exist two constants c_2 and c_3 together with two positive real numbers c_1 and c_4 depending only on d, Ω and θ such that the following inequality holds:*

$$\begin{aligned}
&\eta\|u\|_{\mathrm{L}^2(\Omega)^d}^2 + \nu\|u\|_{\mathcal{D}}^2 + \frac{c_1}{\max\,[\eta,\nu,1/\lambda]}\|p\|_{\mathrm{L}^2(\Omega)}^2 + |p|_{\alpha,\lambda}^2 \leq \\
&\qquad \frac{c_2}{\eta}\|f_1\|_{\mathrm{L}^2(\Omega)^d}^2 + \frac{c_3}{\nu}\|f_2\|_{-1,\mathcal{D}}^2 + c_4\max\,[\eta,\nu,1/\lambda]\ \|g\|_{\mathrm{L}^2(\Omega)}^2
\end{aligned} \tag{33}$$

Remark 4.5 (Dependance of the stability estimate on the stabilization parameter)**.** We note that the bound for the L^2 estimate for the pressure blows up when λ tends to zero, as $|\cdot|_{\alpha,\lambda}$ is a very weak seminorm which vanishes for any constant-by-cluster pressure field: the stabilization of the scheme is thus necessary to control the pressure.

Proof. First, we chose $w \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ such that both estimates of (30) holds. Taking w as a test function in the first relation of (21) yields:

$$\|p\|^2_{\mathrm{L}^2(\Omega)} - \beta_2 |p|^2_{\mathcal{G}} \leq \eta | \int_\Omega u(x) \cdot w(x) \ \mathrm{d}x| + \nu |[u,w]_{\mathcal{D}}| + | \int_\Omega f_1(x) \cdot w(x) \ \mathrm{d}x| + | \int_\Omega f_2(x) \cdot w(x) \ \mathrm{d}x|$$

Using the discrete Poincaré inequality (14) and Young's inequality, we obtain:

$$\begin{aligned} \|p\|^2_{\mathrm{L}^2(\Omega)} - \beta_2 |p|^2_{\mathcal{G}} \quad \leq \quad & \frac{1}{8}\|p\|^2_{\mathrm{L}^2(\Omega)} + 2\eta^2 \mathrm{diam}(\Omega)^2 \beta_1^2 \|u\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{1}{8}\|p\|^2_{\mathrm{L}^2(\Omega)} + 2\nu^2\beta_1^2\|u\|^2_{\mathcal{D}} \\ + \quad & \frac{1}{8}\|p\|^2_{\mathrm{L}^2(\Omega)} + 2\beta_1^2 \mathrm{diam}(\Omega)^2 \|f_1\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{1}{8}\|p\|^2_{\mathrm{L}^2(\Omega)} + 2\beta_1^2 \|f_2\|^2_{-1,\mathcal{D}} \end{aligned} \tag{34}$$

Let ξ_1 and ξ_2 be the positive parameters given by:

$$\xi_1 = \min \left[\frac{1}{8\,\mathrm{diam}(\Omega)^2 \beta_1^2} \ , \ \frac{1}{8\,\beta_1^2} \ , \ \frac{1}{2\,\beta_2} \right] \qquad\qquad \xi_2 = \min\left[\frac{1}{\eta}, \frac{1}{\nu}, \lambda\right] \xi_1 = \frac{1}{\max\,[\eta, \nu, 1/\lambda]}\ \xi_1$$

Note that ξ_1 only depends on θ and Ω. From these definitions, we get by multiplying (34) by ξ_2:

$$\frac{\xi_2}{2}\|p\|^2_{\mathrm{L}^2(\Omega)} - \frac{\lambda}{2}|p|^2_{\mathcal{G}} \leq \frac{\eta}{4}\|u\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{\nu}{4}\|u\|^2_{\mathcal{D}} + \frac{1}{4\eta}\|f_1\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{1}{4\nu}\|f_2\|^2_{-1,\mathcal{D}} \tag{35}$$

Then, taking $v = u$ in the first relation of (21) and $q = p$ in the second one and summing, we obtain, because the discrete gradient is the transposed of the discrete divergence:

$$\eta\|u\|^2_{\mathrm{L}^2(\Omega)^d} + \nu\|u\|^2_{\mathcal{D}} + |p|^2_{\alpha,\lambda} = \int_\Omega f_1(x)\cdot u(x) \ \mathrm{d}x + \int_\Omega f_2(x)\cdot u(x) \ \mathrm{d}x + \int_\Omega g(x) \ p(x) \ \mathrm{d}x$$

By Young's inequality, we then have:

$$\begin{aligned} & \eta\|u\|^2_{\mathrm{L}^2(\Omega)^d} + \nu\|u\|^2_{\mathcal{D}} + |p|^2_{\alpha,\lambda} \leq \\ & \qquad \frac{1}{\eta}\|f_1\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{\eta}{4}\|u\|^2_{\mathrm{L}^2(\Omega)^d} + \frac{1}{\nu}\|f_2\|^2_{-1,\mathcal{D}} + \frac{\nu}{4}\|u\|^2_{\mathcal{D}} + \frac{1}{\xi_2}\|g|^2_{\mathrm{L}^2(\Omega)} + \frac{\xi_2}{4}\|p\|^2_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{36}$$

Summing (35) and (36) and using (23) yields the desired result. □

We can now state the existence and the uniqueness of a discrete solution to (21).

Corollary 4.6 (Existence and uniqueness of a solution to the finite volume scheme)**.** *Under hypotheses* (2)-(4)*, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 3.1. Let $\lambda \in (0, +\infty)$ and $\alpha \leq 1$ be given. We suppose that the following compatibility condition holds (which is nothing more than the compatibility condition associated to the continuous problem):*

$$\int_\Omega g(x) \ \mathrm{d}x = 0$$

Then there exists a unique solution to (21)*.*

Proof. We define the following finite dimensional vector space:

$$V = \{(u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega) \text{ such that } \int_\Omega p \,\mathrm{d}x = 0\}$$

Let F be the linear mapping which associates to $(u,p) \in V$ the pair $(\hat{u},\hat{p})$ defined by the following discrete variational identity:

$$\left|\begin{array}{ll} \eta \displaystyle\int_\Omega u(x)\cdot v(x)\ \mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(v)(x)\ \mathrm{d}x = \int_\Omega \hat{u}(x)\cdot v(x)\ \mathrm{d}x & \forall v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)\ q(x)\ \mathrm{d}x + \langle p,q\rangle_{\alpha,\lambda} = \int_\Omega \hat{p}(x)\ q(x)\ \mathrm{d}x & \forall q \in \mathrm{H}_{\mathcal{D}}(\Omega) \end{array}\right.$$

It is easy to check that this system has a unique solution (choosing as test function the characteristic function of each control volume yields a linear system the matrix of which is the identity). Taking for q the constant function equal to 1 (which lies in $\mathrm{H}_{\mathcal{D}}(\Omega)$) in the last relation, we check by conservativity that the integral of $\hat{p}$ over Ω is zero, which means that $(\hat{u},\hat{p}) \in V$. theorem 4.4 then implies that the kernel of F is reduced to $(0,0)$, which proves that the mapping F is one to one from V onto V. As by assumption the integral of g over Ω is zero, the pair of functions defined by the right hand side of (24) belongs to V, and this concludes the proof. □

4.3. **Estimates of consistency residuals**

We define in this section the consistency residuals appearing in the convergence and error analysis of the scheme and establish the corresponding estimates.

Let v be a function from Ω onto $\mathbb{R}$, the regularity of which will be precised hereafter. For the moment, we only suppose that v is regular enough so that the following definitions make sense. For an internal edge $\sigma = K|L$, we set:

$$\begin{aligned}
R_{\Delta,K|L}(v) &= \frac{1}{d_{K|L}}\ [P_{\mathcal{D}}(v)_L - P_{\mathcal{D}}(v)_K] - \frac{1}{m_{K|L}} \int_{K|L} \nabla v(x)\cdot n_{K|L}\ \mathrm{d}\gamma(x) \\
R_{\mathrm{grad},\mathcal{M},K|L}(v) &= \frac{d_{K,K|L}}{d_{K|L}}\ P_{\mathcal{M}}(v)_K + \frac{d_{L,K|L}}{d_{K|L}}\ P_{\mathcal{M}}(v)_L - \frac{1}{m_{K|L}} \int_{K|L} v(x)\ \mathrm{d}\gamma(x) \\
R_{\mathrm{grad},\mathcal{G},K|L}(v) &= \frac{d_{K,K|L}}{d_{K|L}}\ P_{\mathcal{G}}(v)_K + \frac{d_{L,K|L}}{d_{K|L}}\ P_{\mathcal{G}}(v)_L - \frac{1}{m_{K|L}} \int_{K|L} v(x)\ \mathrm{d}\gamma(x) \\
R_{\mathrm{div},K|L}(v) &= \frac{d_{L,K|L}}{d_{K|L}}\ P_{\mathcal{D}}(v)_K + \frac{d_{K,K|L}}{d_{K|L}}\ P_{\mathcal{D}}(v)_L - \frac{1}{m_{K|L}} \int_{K|L} v(x)\ \mathrm{d}\gamma(x) \\
R_{\mathrm{stab},\mathcal{M},K|L}(v) &= (h_K + h_L)^\alpha\ (P_{\mathcal{M}}(v)_L - P_{\mathcal{M}}(v)_K)
\end{aligned}$$

and, for an external edge σ, $\sigma \in \mathcal{E}(K)$:

$$\begin{aligned}
R_{\Delta,\sigma}(v) &= -\frac{1}{d_{K,\sigma}}\ P_{\mathcal{D}}(v)_K - \frac{1}{m_\sigma} \int_\sigma \nabla v(x)\cdot n_\sigma\ \mathrm{d}\gamma(x) \\
R_{\mathrm{grad},\mathcal{M},\sigma}(v) &= P_{\mathcal{M}}(v)_K - \frac{1}{m_\sigma} \int_\sigma v(x)\ \mathrm{d}\gamma(x) \\
R_{\mathrm{grad},\mathcal{G},\sigma}(v) &= P_{\mathcal{G}}(v)_K - \frac{1}{m_\sigma} \int_\sigma v(x)\ \mathrm{d}\gamma(x) \\
R_{\mathrm{div},\sigma} &= 0 \\
R_{\mathrm{stab},\mathcal{M},\sigma} &= 0
\end{aligned}$$

In addition, we define:

$$R_{\mathrm{o},K}(v) = P_{\mathcal{D}}(v)_K - \frac{1}{m_K}\int_K v(x)\ \mathrm{d}x$$

The consistency residuals of the scheme are now defined as follows:

$$\begin{array}{lll}
\forall v \in \mathrm{H}^2(\Omega)^d,\ R_{\mathrm{o}}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d, & \left(R_{\mathrm{o}}(v)^{(i)}\right)_K = R_{\mathrm{o},K}(v^{(i)}), \quad i=1,\dots,d \\
\forall v \in \mathrm{H}^2(\Omega)^d,\ R_{\Delta}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d, & \left(R_{\Delta}(v)^{(i)}\right)_K = \dfrac{1}{m_K}\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{\Delta,\sigma}(v^{(i)}), \quad i=1,\dots,d \\
\forall v \in \mathrm{L}^2(\Omega),\ R_{\mathrm{grad},\mathcal{M}}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d, & (R_{\mathrm{grad},\mathcal{M}}(v))_K = \dfrac{1}{m_K}\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{\mathrm{grad},\mathcal{M},\sigma}(v)\ n_{K,\sigma} \\
\forall v \in \mathrm{L}^2(\Omega),\ R_{\mathrm{grad},\mathcal{G}}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d, & (R_{\mathrm{grad},\mathcal{G}}(v))_K = \dfrac{1}{m_K}\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{\mathrm{grad},\mathcal{G},\sigma}(v)\ n_{K,\sigma} \\
\forall v \in \mathrm{H}^2(\Omega)^d,\ R_{\mathrm{div}}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega), & (R_{\mathrm{div}}(v))_K = \dfrac{1}{m_K}\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma \left[\sum_{i=1}^d R_{\mathrm{div},\sigma}(v^{(i)})\ e_i\right]\cdot n_\sigma \\
\forall v \in \mathrm{L}^2(\Omega),\ R_{\mathrm{stab},\mathcal{M}}(v) \in \mathrm{H}_{\mathcal{D}}(\Omega), & (R_{\mathrm{stab},\mathcal{M}}(v))_K = \dfrac{1}{m_K}\displaystyle\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{\mathrm{stab},\mathcal{M},\sigma}(v)
\end{array}$$

The following theorem gathers the estimates of the residuals which will be useful in the error analysis.

Theorem 4.7 (Estimates of the consistency residuals)**.** *Let assumption* (2) *hold, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$. *Let $(u,p) \in \mathrm{H}^2(\Omega)^d \cap \mathrm{H}^1_0(\Omega)^d \times \mathrm{H}^1(\Omega)$. Then the following bounds hold:*

$$\|R_\Delta(u)\|_{-1,\mathcal{D}} \leq c_\Delta\ h_{\mathcal{M}}\ |u|_{\mathrm{H}^2(\Omega)^d} \tag{37}$$

$$\|R_{\mathrm{grad},\mathcal{M}}(p)\|_{-1,\mathcal{D}} \leq c_{\mathrm{grad},\mathcal{M}}\ h_{\mathcal{M}}\ |p|_{\mathrm{H}^1(\Omega)} \tag{38}$$

$$\|R_{\mathrm{grad},\mathcal{G}}(p)\|_{-1,\mathcal{D}} \leq c_{\mathrm{grad},\mathcal{G}}\ h_{\mathcal{G}}\ |p|_{\mathrm{H}^1(\Omega)} \tag{39}$$

$$\|R_{\mathrm{o}}(u)\|_{\mathrm{L}^2(\Omega)^d} \leq c_{\mathrm{o}}\ h_{\mathcal{M}}\ \|u\|_{\mathrm{H}^2(\Omega)^d} \tag{40}$$

$$\|R_{\mathrm{stab},\mathcal{M}}(p)\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{stab}}\ h^\alpha_{\mathcal{M}}\ |p|_{\mathrm{H}^1(\Omega)} \tag{41}$$

where c_Δ, $c_{\mathrm{grad},\mathcal{M}}$, $c_{\mathrm{grad},\mathcal{G}}$, c_{o} and c_{stab} only depend on d, Ω and θ.

If in addition the mesh is super–admissible in the sense of definition 3.1, then:

$$\|R_{\mathrm{div}}(u)\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{div}}\ h_{\mathcal{M}}\ |u|_{\mathrm{H}^2(\Omega)^d} \tag{42}$$

where c_{div} only depends on d, Ω and θ.

Proof.

Step 1: Proof of the $\mathrm{H}^{-1}_{\mathcal{D}}$ estimates (37)–(39). The proof of the three discrete H^{-1} estimates (37), (38) and (39) being similar, we shall give the general idea and then apply it only to obtain (39). Consider a consistency residual $R \in \mathrm{H}_{\mathcal{D}}(\Omega)$ under the general form:

$$R_K = \frac{1}{m_K}\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{K,\sigma}$$

with, for any internal edge $\sigma = K|L$, $R_{K,\sigma} = -R_{L,\sigma}$, and we define $R_\sigma = |R_{K,\sigma}|$. Let v be a function of $\mathrm{H}_{\mathcal{D}}$. Then we have:

$$\int_\Omega R(x)\ v(x)\ \mathrm{d}x = \sum_{K\in\mathcal{M}} m_K R_K v_K = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{K,\sigma}$$

Reordering the sums, we get:

$$\int_\Omega R(x)\ v(x)\ \mathrm{d}x = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma R_{K,\sigma}(v_K - v_L) + \sum_{\sigma\in\mathcal{E}_{\rm ext},\ \sigma\in\mathcal{E}(K)} m_\sigma R_{K,\sigma} v_K$$

and, by the (discrete) Cauchy-Schwarz inequality:

$$\begin{aligned} |\int_\Omega R(x)\ v(x)\ \mathrm{d}x| &\leq \left[\sum_{\sigma\in\mathcal{E}} d_\sigma m_\sigma R_\sigma^2\right]^{1/2} \left[\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} \frac{m_\sigma}{d_\sigma}(v_K - v_L)^2 + \sum_{\sigma\in\mathcal{E}_{\rm ext},\ \sigma\in\mathcal{E}(K)} \frac{m_\sigma}{d_\sigma} v_K^2\right]^{1/2} \\ &\leq \left[\sum_{\sigma\in\mathcal{E}} d_\sigma m_\sigma R_\sigma^2\right]^{1/2} \|v\|_{\mathcal{D}}. \end{aligned}$$

In order to obtain an estimate in the $\|\cdot\|_{-1,\mathcal{D}}$ norm of R, there only remains to bound the sum $\sum_{\sigma\in\mathcal{E}} d_\sigma m_\sigma R_\sigma^2$, which can be done thanks to the elementary residual bounds stated in lemma A.2 or corollary A.4. As already mentioned, we only give here an exemple of application of this technique to the estimate of $R_{\mathrm{grad},\mathcal{G}}$. Let p be a function of $\mathrm{H}^1(\Omega)$ and v a function of $\mathrm{H}_{\mathcal{D}}(\Omega)^d$. By definition, we have:

$$\int_\Omega R_{\mathrm{grad},\mathcal{G}}(p)(x)\cdot v(x)\ \mathrm{d}x = \sum_{K\in\mathcal{M}} m_K \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}(K)} m_\sigma R_{\mathrm{grad},\mathcal{G},\sigma}(p)\ n_{K,\sigma}\cdot v_K$$

By the computation described above, we thus get:

$$\begin{aligned} |\int_\Omega R_{\mathrm{grad},\mathcal{G}}(p)(x)\cdot v(x)\ \mathrm{d}x| \leq &\left[\sum_{\sigma\in\mathcal{E}_{\rm int}\cup\mathcal{E}_{\rm ext}} d_\sigma m_\sigma R_{\mathrm{grad},\mathcal{G},\sigma}(p)^2\right]^{1/2} \\ &\left[\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} \frac{m_\sigma}{d_\sigma}[(v_K - v_L)\cdot n_{K,\sigma}]^2 + \sum_{\sigma\in\mathcal{E}_{\rm ext},\ \sigma\in\mathcal{E}(K)} \frac{m_\sigma}{d_\sigma}[v_K\cdot n_{K,\sigma}]^2\right]^{1/2} \end{aligned}$$

Now, by the Cauchy-Schwarz inequality, $\forall a\in\mathbb{R}^d$, $a\cdot n \leq (a^2)^{1/2}$, so that:

$$|\int_\Omega R_{\mathrm{grad},\mathcal{G}}(p)(x)\cdot v(x)\ \mathrm{d}x| \leq \left[\sum_{\sigma\in\mathcal{E}_{\rm int}\cup\mathcal{E}_{\rm ext}} d_\sigma m_\sigma R_{\mathrm{grad},\mathcal{G},\sigma}(p)^2\right]^{1/2} \|v\|_{\mathcal{D}}$$

Using the elementary residual estimate (77) of Corollary A.4 in the Appendix yields:

$$|\int_\Omega R_{\mathrm{grad},\mathcal{G}}(p)(x)\cdot v(x)\ \mathrm{d}x| \leq c\ \|v\|_{\mathcal{D}} \left[\sum_{\sigma\in\mathcal{E}_{\rm int}\cup\mathcal{E}_{\rm ext}} d_\sigma m_\sigma \frac{h_{\mathcal{D}}}{m_\sigma}\ |p|^2_{\mathrm{H}^1(C_\sigma)}\right]^{1/2}$$

where $C_\sigma = C_{G_K}\cup C_{G_L}$ if $\sigma\in\mathcal{E}_{\rm int}, \sigma=K|L$ and $C_\sigma = C_{G_K}$ if $\sigma\in\mathcal{E}_{\rm int}, \sigma\in\mathcal{E}(K)$. Thanks to the regularity assumption on the mesh, for a given control volume K, the number of domains C_σ including K is bounded by a constant c_σ, the $|p|^2_{\mathrm{H}^1(K)}$ also appears in the above summation only c_σ times and we obtain:

$$|\int_\Omega R_{\mathrm{grad},\mathcal{G}}(p)(x)\cdot v(x)\ \mathrm{d}x| \leq c\ \|v\|_{\mathcal{D}}\ c_\sigma\ (\max_{\sigma\in\mathcal{E}_{\rm int}\cup\mathcal{E}_{\rm ext}} d_\sigma\ h_{\mathcal{D}})^{1/2}\ |p|_{\mathrm{H}^1(\Omega)}$$

which yields the desired estimate for $R_{\mathrm{grad},\mathcal{G}}$ and concludes step 1.

Step 2: Proof of (40)–(41). Estimates of the consistency residuals in L^2 are obtained in a straightforward way from the bounds of the elementary consistency residuals estimates of lemma A.2 or Corollary A.4 in the Appendix. As an exemple, we detail here the bound (42); the bounds (40) and (41) are obtained in a similar way.

Let $v(.)$ be a function of $H^2(\Omega)^d$, and assume the mesh to be super–admissible. By definition, we have:

$$\|R_{\rm div}(u)\|^2_{L^2(\Omega)} = \sum_{K\in\mathcal{M}} m_K \left[\frac{1}{m_K} \sum_{\sigma\in\mathcal{E}(K)} m_\sigma \left[\sum_{i=1}^d R_{{\rm div},\sigma}(v^{(i)})\ e_i \right] \cdot n_{K,\sigma} \right]^2$$

Developping the sum, we have:

$$\left[\sum_{\sigma\in\mathcal{E}(K)} m_\sigma \left[\sum_{i=1}^d R_{{\rm div},\sigma}(v^{(i)})\ e_i \right] \cdot n_{K,\sigma} \right]^2 \leq c \sum_{\sigma\in\mathcal{E}(K)} m_\sigma^2 R_{{\rm div},\sigma}(v^{(i)})^2$$

where c depends only on the number of edges of a control volume and on the space dimension d. Since, by definition, $R_{{\rm div},\sigma}$ vanishes on external boundaries, we then get:

$$\|R_{\rm div}(u)\|^2_{L^2(\Omega)} \leq c \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} (\frac{1}{m_K} + \frac{1}{m_L})\ m_\sigma^2 \sum_{i=1}^d R_{{\rm div},\sigma}(v^{(i)})^2$$

The bound (69) thus yields:

$$\|R_{\rm div}(u)\|^2_{L^2(\Omega)} \leq c \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} (\frac{1}{m_K} + \frac{1}{m_L})\ m_\sigma^2 \sum_{i=1}^d \frac{[\displaystyle\sum_{\sigma\in\mathcal{E}(K)} h_K]^4}{m_\sigma {\rm d}_\sigma}\ |v^{(i)}|^2_{H^2(D_\sigma)}$$

The estimate of $R_{\rm div}(v)$ is then completed using the regularity assumptions for the mesh.

□

4.4. Convergence of the scheme

The aim of this section is to prove the convergence of the scheme (21) to the unique solution of the generalized Stokes problem without any regularity assumption for this latter. This result is stated in the following theorem.

Theorem 4.8 (Convergence for the generalized Stokes problem). *Under hypotheses* (2)-(4)*, let* $(\bar u, \bar p)$ *be the unique weak solution of the Stokes problem* (8) *in the sense of definition 2.2. Let* $\theta > 0$ *be given and let* $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ *be a sequence of admissible discretizations of* Ω *in the sense of definition 3.1 such that, for all* $m \in \mathbb{N}$, ${\rm regul}(\mathcal{D}^{(m)}) \geq \theta$ *and such that* $\lim_{m\to\infty} h_{\mathcal{D}}^{(m)} = 0$*. We denote by* $(u^{(m)}, p^{(m)}) \in H_{\mathcal{D}^{(m)}}(\Omega)^d \times H_{\mathcal{D}^{(m)}}(\Omega)$ *the unique solution to* (21) *with the discretization* $\mathcal{D}^{(m)}$*, with two given parameters* $\lambda \in \mathbb{R}_+^\star$ *and* $\alpha \leq 1$ *independent of* m*. Then the following holds:*

1. *the sequence* $(u^{(m)})_{m\in\mathbb{N}}$ *converges to* $\bar u$ *in* $(L^2(\Omega))^d$ *and* $(p^{(m)})_{m\in\mathbb{N}}$ *weakly converges to* $\bar p$ *in* $L^2(\Omega)$,
2. $(u^{(m)})_{m\in\mathbb{N}}$ *and* $(p^{(m)})_{m\in\mathbb{N}}$ *satisfies the following additional convergence results:*

$$\lim_{m\to\infty} [u^{(m)}, u^{(m)}]_{\mathcal{D}} = \int_\Omega \nabla \bar u(x) : \nabla \bar u(x)\ {\rm d}x \tag{43}$$

and:

$$\lim_{m\to\infty} |p^{(m)}|_{\mathcal{D},\lambda} = 0 \tag{44}$$

Remark 4.9. Thanks to item 2 of the above proposition, we get that the discrete gradient, defined from u by the definitions given in [10] or [12], converges in $(\mathrm{L}^2(\Omega)^d)^d$ to $\nabla \bar{u}$.

Proof.

Proof of item(1). We first prove, under the hypotheses and with the notations of the above theorem, the existence of a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ such that the corresponding sequence $(u^{(m)})_{m\in\mathbb{N}}$ converges in $(\mathrm{L}^2(\Omega))^2$ to $\bar{u}$ and the sequence $(p^{(m)})_{m\in\mathbb{N}}$ weakly converges in $(\mathrm{L}^2(\Omega))^2$ to $\bar{p}$, as $m \to \infty$; then this convergence property will hold for the whole sequence thanks to the uniqueness of $(\bar{u}, \bar{p})$, weak solution to the generalized Stokes problem, in the sense of definition 2.2.

By the fact that $\|u^{(m)}\|_{\mathcal{D}}$ is bounded independently of m by the stability estimate (33), we obtain (see [9, lemma 9.3, p. 770]) the following estimate on the translates of $u^{(m)}$. For all $m \in \mathbb{N}$, there exists c, only depending on d, Ω, ν, f, g and θ such that:

$$\int_\Omega \left[u^{(m,k)}(x+\xi) - u^{(m,k)}(x)\right]^2 \mathrm{d}x \le c\, |\xi| \left[|\xi| + 4h^{(m)}\right], \qquad \text{for } k = 1, \ldots, d, \quad \forall \xi \in \mathbb{R}^d, \tag{45}$$

where $u^{(m,k)}$ denotes the k-th component of $u^{(m)}$ and $h^{(m)}$ stands for the size of the discretization $\mathcal{D}^{(m)}$. We then apply Kolmogorov's theorem, and obtain the existence of a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ and of $\bar{u} \in \mathrm{H}^1_0(\Omega)^d$ such that $(u^{(m)})_{m\in\mathbb{N}}$ converges to $\bar{u}$ in $\mathrm{L}^2(\Omega)^d$.

In addition, thanks to the fact that $\|p^{(m)}\|_{\mathrm{L}^2(\Omega)}$ is bounded independently of m by the same bound (33), we extract from this subsequence another one (still denoted $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$) such that $(p^{(m)})_{m\in\mathbb{N}}$ weakly converges to some function $\bar{p}$ in $\mathrm{L}^2(\Omega)$.

In order prove item (1), *i.e.* the convergence of the scheme, we now must show that $(\bar{u}, \bar{p})$ is the solution of (9). By density, it is sufficient to prove that this variational problem is satisfied for any test fuction in $C_c^\infty(\Omega)^d$. This will be proved by passing to the limit in the scheme. We thus take $\varphi \in C_c^\infty(\Omega)^d$, and suppose that m is large enough and thus $h_{\mathcal{D}}^{(m)}$ is small enough to ensure that, for all $K \in \mathcal{M}$ such that $K\cap$ support$(\varphi) \neq \emptyset$, $\partial K \cap \partial\Omega = \emptyset$. Let us take $v = P_{\mathcal{D}^{(m)}}\varphi$ in (21), which yields:

$$\eta \int_\Omega u^{(m)}(x) \cdot P_{\mathcal{D}^{(m)}}\varphi(x)\ \mathrm{d}x + \nu[u^{(m)}, P_{\mathcal{D}^{(m)}}\varphi]_{\mathcal{D}} - \int_\Omega p^{(m)}(x)\ \mathrm{div}_{\mathcal{D}^{(m)}}(P_{\mathcal{D}^{(m)}}\varphi)(x)\ \mathrm{d}x = \int_\Omega f(x) \cdot P_{\mathcal{D}^{(m)}}\varphi(x)\ \mathrm{d}x$$

We write this latter relation as follows:

$$\eta \int_\Omega u^{(m)}(x) \cdot \varphi(x)\ \mathrm{d}x - \nu \int_\Omega u^{(m)} \Delta\varphi\ \mathrm{d}x - \int_\Omega p^{(m)}(x)\ \mathrm{div}\varphi(x)\ \mathrm{d}x + R_1 = \int_\Omega f(x) \cdot \varphi(x)\ \mathrm{d}x + R_2 \tag{46}$$

where R_1 is the sum of three terms $R_1 = T_1 + T_2 + T_3$ which are defined hereafter, together with R_2. We first have:

$$T_1 = \eta \int_\Omega u^{(m)}(x) \cdot (P_{\mathcal{D}^{(m)}}\varphi(x) - \varphi(x))\ \mathrm{d}x$$

and thus, by the Cauchy-Schwarz inequality and by theorem 4.7:

$$|T_1| \le \eta\ \|u^{(m)}\|_{\mathrm{L}^2(\Omega)^d}\ \|R_{\mathrm{o}}(\varphi)\|_{\mathrm{L}^2(\Omega)^d} \le c_1(\varphi)\ h^{(m)}\ \|u^{(m)}\|_{\mathrm{L}^2(\Omega)^d}$$

Using as in equation (24) the classical finite volume expression of the diffusion term, T_2 reads:

$$T_2^{(k)} = \nu \sum_{K\in\mathcal{M}} (P_{\mathcal{D}^{(m)}}\varphi)_K \sum_{\sigma\in\mathcal{E}(K),\ \sigma=K|L} \frac{m_\sigma}{d_\sigma}\ (u_K^{(m,k)} - u_L^{(m,k)}) + \sum_{K\in\mathcal{M}} u_K^{(m,k)} \int_K \Delta\varphi^{(k)}(x)\ \mathrm{d}x$$

Reordering the sums, we get:

$$T_2^{(k)} = \nu \sum_{K\in\mathcal{M}} m_K u_K^{(m,k)} \sum_{\sigma\in\mathcal{E}(K),\ \sigma=K|L} R_{\mathrm{div},K|L}(\varphi^{(k)})$$

and thus, by the definition of the $\|\cdot\|_{-1,\mathcal{D}}$ norm then theorem 4.7:

$$|T_2| \leq \nu\ \|u^{(m)}\|_{\mathcal{D}^{(m)}}\ \|R_{\mathrm{div}}(\varphi)\|_{-1,\mathcal{D}^{(m)}} \leq c_2(\varphi)\ h^{(m)}\ \|u^{(m)}\|_{\mathcal{D}}$$

The third term is defined and bounded as follows:

$$|T_3| = |\int_\Omega p^{(m)}(x)\ R_{\mathrm{div}}(\varphi)(x)\ \mathrm{d}x| \leq c_3(\varphi)\ h^{(m)}\ \|p^{(m)}\|_{\mathrm{L}^2(\Omega)}$$

and, finally:

$$|R_2| = |\int_\Omega f(x)\ R_{\mathrm{o}}(\varphi)(x)\ \mathrm{d}x| \leq c_4(\varphi)\ h^{(m)}\ \|f\|_{\mathrm{L}^2(\Omega)}$$

We thus obtain that both R_1 and R_2 tend to zero when m tends to ∞, and passing to the limit in (46) yields that the first equation of the generalized Stokes problem is satisfied. The last step to prove that $(\bar{u},\bar{p})$ is the weak solution of the Stokes problem is to show that $\mathrm{div}(\bar{u})(x) = g(x)$ for a.e. $x\in\Omega$. Let us take now $\varphi\in C_c^\infty(\Omega)$ and $q = P_{\mathcal{G}^{(m)}}\varphi$ in the second equation of (21), to obtain:

$$\int_\Omega (\mathrm{div}_{\mathcal{D}} u^{(m)})(x)\ (P_{\mathcal{G}^{(m)}}\varphi)(x)\ \mathrm{d}x + \langle p^{(m)}, P_{\mathcal{G}^{(m)}}\varphi\rangle_{\alpha,\lambda} = \int_\Omega g(x)\ (P_{\mathcal{G}^{(m)}}\varphi)(x)\ \mathrm{d}x$$

As $P_{\mathcal{G}^{(m)}}\varphi$ is constant over each cluster, the stabilization term vanishes. Using the fact that the discrete divergence is the transposed of the discrete gradient, we get:

$$\int_\Omega (\mathrm{div}_{\mathcal{D}} u^{(m)})(x)\ (P_{\mathcal{G}^{(m)}}\varphi)(x)\ \mathrm{d}x = -\int_\Omega u^{(m)}(x)\cdot(\nabla_{\mathcal{D}} P_{\mathcal{G}^{(m)}}\varphi)(x)\ \mathrm{d}x = -\int_\Omega u^{(m)}(x)\cdot(\nabla\varphi)(x) + R$$

where R reads:

$$R = \sum_{K\in\mathcal{M}} u_K^{(m)}\cdot \sum_{\sigma\in\mathcal{E}(K),\ \sigma=K|L} m_\sigma\ R_{\mathrm{grad},\mathcal{G},\sigma}(\varphi)\ n_\sigma = \int_\Omega u_K^{(m)}\cdot R_{\mathrm{grad},\mathcal{G}}(\varphi)(x)\ \mathrm{d}x$$

By the Cauchy-Schwarz inequality and the estimate of theorem 4.7, we thus have:

$$|R| \leq c(\varphi)\ h^{(m)}\ \|u^{(m)}\|_{\mathrm{L}^2(\Omega)}$$

As the convergence of $\int_\Omega g(x)\ (P_{\mathcal{G}^{(m)}}\varphi)(x)\ \mathrm{d}x$ to $\int_\Omega g(x)\varphi)(x)\ \mathrm{d}x$ is easily seen, this concludes the proof of item (1) of the above theorem.

Proof of item (2). Setting $v = u^{(m)}$ in the first relation of (21) and $q = p^{(m)}$ in the second one gives:

$$\eta\int_\Omega u^{(m)}(x)^2\ \mathrm{d}x + \nu\|u^{(m)}\|^2_{\mathcal{D}^{(m)}} + |p^{(m)}|^2_{\mathcal{D}^{(m)},\lambda} = \int_\Omega f(x)\cdot u^{(m)}(x)\ \mathrm{d}x + \int_\Omega g(x)\ p^{(m)}(x)\ \mathrm{d}x$$

Passing to the limit $m\to\infty$ in the above equation yields:

$$\eta\int_\Omega \bar{u}(x)^2\ \mathrm{d}x + \limsup_{m\to\infty}\left[\nu\|u^{(m)}\|^2_{\mathcal{D}^{(m)}} + |p^{(m)}|^2_{\mathcal{D}^{(m)},\lambda}\right] \leq \int_\Omega f(x)\cdot\bar{u}(x)\ \mathrm{d}x + \int_\Omega g(x)\ \bar{p}(x)\ \mathrm{d}x$$

Since we have, from the corresponding choice for the test functions in the continuous problem (9):

$$\eta \int_\Omega \bar u(x)^2 \ \mathrm{d}x + \nu \int_\Omega \nabla \bar u(x) : \nabla \bar u(x) \ \mathrm{d}x = \int_\Omega f(x) \cdot \bar u(x) \ \mathrm{d}x + \int_\Omega g(x) \ \bar p(x) \ \mathrm{d}x$$

and from the results of [19, lemma 2.2]:

$$\int_\Omega \nabla \bar u(x) : \nabla \bar u(x) \ \mathrm{d}x \leq \liminf_{m\to\infty} \|u^{(m)}\|^2_{\mathcal{D}^{(m)}}$$

we get:

$$\limsup_{m\to\infty} \left[\nu \|u^{(m)}\|^2_{\mathcal{D}^{(m)}} + |p^{(m)}|^2_{\mathcal{D}^{(m)},\lambda} \right] \leq \nu \int_\Omega \nabla \bar u(x) : \nabla \bar u(x) \ \mathrm{d}x \leq \liminf_{m\to\infty} \|u^{(m)}\|^2_{\mathcal{D}^{(m)}}$$

which yields (43) and (44). □

4.5. **Error analysis**

Theorem 4.10 (Error estimate)**.** *We suppose that hypotheses* (2)-(4) *hold. Let* $\mathcal{D}$ *be an admissible discretization of* Ω *in the sense of definition 3.1 and let* $\theta > 0$ *be such that* $\mathrm{regul}(\mathcal{D}) > \theta$. *Let* $\lambda \in (0, +\infty)$ *and* $\alpha \leq 1$ *be given. We suppose that the solution of the continuous problem* (8),$(\bar u, \bar p)$, *lies in* $(\mathrm{H}^2(\Omega)^d \cap \mathrm{H}^1_0(\Omega)^d) \times \mathrm{H}^1(\Omega)$. *Let* $(u, p) \in \mathrm{H}_\mathcal{D}(\Omega)^d \times \mathrm{H}_\mathcal{D}(\Omega)$ *be a solution to* (21). *Then, if* $\alpha = 1$, *there exists a positive real number* c *depending only on* Ω *and* θ *such that the following inequality holds:*

$$\begin{aligned} &\eta \|u - P_\mathcal{D}(\bar u)\|^2_{\mathrm{L}^2(\Omega)^d} + \nu \|u - P_\mathcal{D}(\bar u)\|^2_\mathcal{D} + \frac{c_1}{\max[\eta, \nu, 1/\lambda]} \|p - P_\mathcal{M}(\bar p)\|^2_{\mathrm{L}^2(\Omega)} \\ &\qquad \leq c \ h^2_\mathcal{M} \ \left[(1 + \max[\eta, \nu, 1/\lambda]) \ \|\bar u\|_{\mathrm{H}^2(\Omega)^d} + (\frac{1}{\nu} + \lambda \max[\eta, \nu, 1/\lambda]) \ |\bar p|_{\mathrm{H}^1(\Omega)} \right] \end{aligned} \tag{47}$$

If $\alpha < 1$, *the same estimate holds with* $h_\mathcal{M}$ *replaced by* $h_\mathcal{G}$, i.e. *there exists* c *only depending on* d, Ω *and* θ *such that:*

$$\begin{aligned} &\eta \|u - P_\mathcal{D}(\bar u)\|^2_{\mathrm{L}^2(\Omega)^d} + \nu \|u - P_\mathcal{D}(\bar u)\|^2_\mathcal{D} + \frac{c_1}{\max[\eta, \nu, 1/\lambda]} \|p - P_\mathcal{G}(\bar p)\|^2_{\mathrm{L}^2(\Omega)} \\ &\qquad \leq c \ h^2_\mathcal{G} \ \left[(1 + \max[\eta, \nu, 1/\lambda]) \ \|\bar u\|_{\mathrm{H}^2(\Omega)^d} + \frac{1}{\nu} \ |\bar p|_{\mathrm{H}^1(\Omega)} \right] \end{aligned} \tag{48}$$

Proof. Case $\alpha = 1$

We define $e \in \mathrm{H}_\mathcal{D}(\Omega)^d$ and $\epsilon \in \mathrm{H}_\mathcal{D}(\Omega)$ by $e_K = u_K - P_\mathcal{D}(\bar u)_K$ and $\epsilon_K = p_K - P_\mathcal{M}(\bar p)_K$ Substracting the same terms at the left and right hand side of the discrete momentum balance equation, we get, for each control volume K of $\mathcal{M}$:

$$\begin{aligned} &\eta \ m_K e_K - \nu \sum_{L\in\mathcal{N}_K} \frac{m_\sigma}{d_\sigma} \ (e_L - e_K) - \nu \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{m_\sigma}{d_{K,\sigma}} \ (-e_K) + \sum_{L\in\mathcal{N}_K} m_\sigma \ \frac{d_{L,\sigma}}{d_\sigma} \ (\epsilon_L - \epsilon_K) \ n_\sigma = \\ &\quad \eta \ m_K P_\mathcal{D}(\bar u)_K + \nu \sum_{L\in\mathcal{N}_K} \frac{m_\sigma}{d_\sigma} \ (P_\mathcal{D}(\bar u)_L - P_\mathcal{D}(\bar u)_K) + \nu \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{m_\sigma}{d_{K,\sigma}} \ (-p_\mathcal{D}(\bar u)_K) \\ &\quad - \sum_{L\in\mathcal{N}_K} m_\sigma \ \left[\frac{d_{K,\sigma}}{d_\sigma} \ P_\mathcal{M}(\bar p)_K + \frac{d_{L,\sigma}}{d_\sigma} \ P_\mathcal{M}(\bar p)_L \right] \ n_\sigma + \int_K f(x) \ \mathrm{d}x \end{aligned}$$

The regularity of $\bar u$ and $\bar p$ assumed in the statement of the theorem allows to integrate the continuous partial derivative equation (8) over each element K:

$$\eta \int_K \bar u(x)\ \mathrm{d}x + \nu \int_{\partial K} -\nabla \bar u(x)\cdot n\ \mathrm{d}\gamma(x) + \int_{\partial K} \bar p(x)\ n\ \mathrm{d}\gamma(x) = \int_K f(x)\ \mathrm{d}x$$

Substracting this relation to the previous one, we get, for each control volume K of $\mathcal{M}$:

$$\eta\ m_K e_K - \nu \sum_{L\in\mathcal{N}_K} \frac{m_\sigma}{d_\sigma}\ (e_L - e_K) - \nu \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{m_\sigma}{d_{K,\sigma}}\ (-e_K) + \sum_{L\in\mathcal{N}_K} m_\sigma\ \frac{d_{L,\sigma}}{d_\sigma}\ (\epsilon_L - \epsilon_K)\ n_\sigma = \eta \int_K R_{\mathrm{o}}(\bar u)(x)\ \mathrm{d}x + \nu \int_K R_\Delta(\bar u)(x)\ \mathrm{d}x + \int_K R_{\mathrm{grad},\mathcal{M}}(\bar p)(x)\ \mathrm{d}x$$

Repeating the same process with the mass balance equation yields, once again for each control volume K of $\mathcal{M}$:

$$\sum_{L\in\mathcal{N}_K} m_\sigma\ (\frac{d_{L,\sigma}}{d_\sigma}\ e_K + \frac{d_{K,\sigma}}{d_\sigma}\ e_L)\cdot n_\sigma - \lambda \sum_{L\in\mathcal{N}_K\cap G_K} (h_K + h_L)\ m_\sigma\ (\epsilon_L - \epsilon_K) = \int_K R_{\mathrm{div}}(\bar u)(x)\ \mathrm{d}x + \lambda \int_K R_{\mathrm{stab},\mathcal{M}}(\bar p)(x)\ \mathrm{d}x$$

The result then follows by combining the estimate of the consistency residuals (theorem 4.7) with the stability result of theorem 4.4, with the following choice:

$$f_1 = \eta\, R_{\mathrm{o}}(\bar u), \qquad f_2 = \nu\, R_\Delta(\bar u) + R_{\mathrm{grad},\mathcal{M}}(\bar p), \qquad g = R_{\mathrm{div}}(\bar u) + \lambda\, R_{\mathrm{stab},\mathcal{M}}(\bar p)$$

Case $\alpha < 1$

The proof for the case $\alpha < 1$ follows strictly the same line, replacing $P_{\mathcal{M}}(\bar p)$ by $P_{\mathcal{G}}(\bar p)$, so that the definition of the pressure error becomes $\epsilon_K = p_K - P_{\mathcal{G}}(\bar p)_K,\ \forall K \in \mathcal{M}$. We remark that the stabilization terms vanish when applied to $P_{\mathcal{G}}(\bar p)$. The system of equations governing the errors is thus formally the same as in the case $\alpha = 1$, excepting for the stabilization residual which disappears, and the conclusion once again follows from theorems 4.4 and 4.7. □

5. The finite volume scheme for the Navier-Stokes equations

We first present here the scheme used to solve Navier-Stokes equations (section 5.1), then the remaining of the section is devoted to its analysis. Compared to the Stokes problem considered in the previous section, Navier-Stokes equations introduce additional difficulties: indeed, because of the nonlinear term in the momentum balance equation, we no longer get a uniform estimate on the pressure with respect to the mesh size as in the linear case. However, a first rough bound for the nonlinear term allows to get a weak estimate on the pressure, *i.e.* a bound which may blow up as the mesh size tends to 0; this is sufficient to prove the existence of the discrete solution to the scheme (section 5.2). Next, the convergence study is rather tricky because the same lack of estimate on the pressure prevents to control the pressure gradient term when tested against the standard interpolation of a divergence-free regular function; the idea of the proof of convergence is then to use the discrete Stokes problem to perform this interpolation. This technique yields the strong convergence (in $\mathrm{L}^2(\Omega)^d$) of the velocity (section 5.3).

5.1. The finite volume scheme

As in the section devoted to the generalized Stokes problem, we first write the finite volume scheme under consideration in a variational-like setting. Under hypotheses (2)-(4) and $\mathcal{D}$ being an admissible discretization of Ω in the sense of definition 3.1, we look for (u,p) such that:

$$\left|\begin{array}{ll} (u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\ \mathrm{d}x = 0, & \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\,\mathrm{d}x + b_{\mathcal{D}}(u,u,v) + \nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\,\mathrm{div}_{\mathcal{D}}(v)(x)\,\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\,\mathrm{d}x & \forall v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)\ q(x)\ \mathrm{d}x + \langle p,q\rangle_{\mathcal{D},\lambda} = 0 & \forall q \in \mathrm{H}_{\mathcal{D}}(\Omega) \end{array}\right. \tag{49}$$

where λ is a positive real and, for $u,v,w \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$, we define the following approximation for $b(u,v,w)$:

$$b_{\mathcal{D}}(u,v,w) = \frac{1}{2}\sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma \left[\frac{d_{L,\sigma}u_K + d_{K,\sigma}u_L}{d_\sigma}\cdot n_\sigma\right]\ [v_L\cdot w_K] \tag{50}$$

Using the discrete mass balance to transform the expression of the convective term, system (49) is equivalent to finding the family of vectors $(u_K)_{K\in\mathcal{M}} \subset \mathbb{R}^d$, and scalars $(p_K)_{K\in\mathcal{M}} \subset \mathbb{R}$ solution of the system of equations obtained by writing for each control volume K of $\mathcal{M}$:

$$\left|\begin{array}{l} \eta\ m_K\ u_K - \nu \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma}(u_L-u_K) - \nu \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}(K)} \frac{m_\sigma}{d_{K,\sigma}}(0-u_K) \\ + \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma \left[\frac{d_{L,\sigma}u_K + d_{K,\sigma}u_L}{d_\sigma}\cdot n_\sigma\right] \frac{u_K+u_L}{2} - \lambda\left[\sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}\ (h_K+h_L)^\alpha(p_L-p_K)\right]\frac{u_K}{2} \\ \qquad + \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma\ \frac{d_{L,\sigma}p_L + d_{K,\sigma}p_K}{d_{K|L}}\ n_{K|L} + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}(K)} m_\sigma\ p_K\ n_{K,\sigma} = \int_K f(x)\ \mathrm{d}x \\ \displaystyle\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma\ \frac{d_{L,\sigma}u_K + d_{K,\sigma}u_L}{d_\sigma}\cdot n_\sigma - \lambda \sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}\ (h_K+h_L)^\alpha(p_L-p_K) = 0 \end{array}\right.$$

supplemented by the relation:

$$\sum_{K\in\mathcal{M}} m_K\ p_K = 0$$

The above scheme is written in a conservative form except for the second term in the discretisation of the trilinear form, *i.e.*:

$$\lambda\left[\sum_{L\in\mathcal{N}_K\cap G_K} m_{K|L}\ (h_K+h_L)^\alpha(p_L-p_K)\right]\frac{u_K}{2}$$

which stems from the stabilization term.

5.2. Stability and existence of discrete solutions

Let us first remark that $b_{\mathcal{D}}(\cdot,\cdot,\cdot)$ is such that, for all $u,v\in \mathrm{H}_{\mathcal{D}}(\Omega)^d$:

$$b_{\mathcal{D}}(u,v,v)=0. \tag{51}$$

Let us also remark that, in a similar way as in [13], we have the existence of c_b, only depending on d and Ω, such that:

$$|b_{\mathcal{D}}(u,v,w)|\le c_b\ \|u\|_{\mathcal{D}}\|v\|_{\mathcal{D}}\|w\|_{\mathcal{D}},\ \forall u,v,w\in \mathrm{H}_{\mathcal{D}}(\Omega)^d. \tag{52}$$

Lemma 5.1 (Discrete $\mathrm{H}^1_0(\Omega)$ estimate on the velocities)**.** *Under hypotheses* (2)-(4)*, let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1, $\lambda\in(0,+\infty)$ and $\alpha\le 1$ be given. For $\rho\in[0,1]$, we assume that (u,p) is a solution to the following system of equations (which reduces to* (49) *as $\rho=1$ and to* (21) *as $\rho=0$):*

$$\left|\begin{array}{ll}(u,p)\in \mathrm{H}_{\mathcal{D}}(\Omega)^d\times \mathrm{H}_{\mathcal{D}}(\Omega)\ \textit{with}\ \displaystyle\int_\Omega p(x)\ \mathrm{d}x=0, & \\[2ex] \eta\displaystyle\int_\Omega u(x)\cdot v(x)\,\mathrm{d}x+\rho\ b_{\mathcal{D}}(u,u,v)+\nu[u,v]_{\mathcal{D}} & \\ \qquad\qquad -\displaystyle\int_\Omega p(x)\,\mathrm{div}_{\mathcal{D}}(v)(x)\,\mathrm{d}x=\int_\Omega f(x)\cdot v(x)\,\mathrm{d}x & \forall v\in \mathrm{H}_{\mathcal{D}}(\Omega)^d\\[2ex] \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)\ q(x)\ \mathrm{d}x+\langle p,q\rangle_{\mathcal{D},\lambda}=0 & \forall q\in \mathrm{H}_{\mathcal{D}}(\Omega)\end{array}\right. \tag{53}$$

Then u and p satisfy the following estimates:

$$\nu\|u\|_{\mathcal{D}}\ \le\ \mathrm{diam}(\Omega)\ \|f\|_{\mathrm{L}^2(\Omega)^d} \tag{54}$$
$$\nu\ |p|^2_{\mathcal{D},\lambda}\ \le\ \mathrm{diam}(\Omega)^2\ \|f\|^2_{\mathrm{L}^2(\Omega)^d} \tag{55}$$

Moreover, there exist $c_{\mathrm{p},\mathcal{D}}$ only depending on d, Ω, η, ν, λ, f and $\mathcal{D}$ and not on $\rho\in[0,1]$, such that the following inequality holds:

$$\|p\|_{\mathrm{L}^2(\Omega)}\le c_{\mathrm{p},\mathcal{D}} \tag{56}$$

Proof. The proof of (54) and (55) is first obtained by setting $(v,q)=(u,p)$ in (53) and using the property (51) on the discrete form $b_{\mathcal{D}}(\cdot,\cdot,\cdot)$. We then consider the function $\widetilde{f}\in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ defined by:

$$\int_\Omega \widetilde{f}(x)\cdot v(x)\ \mathrm{d}x=\int_\Omega f(x)\cdot v(x)\ \mathrm{d}x-\rho\ b_{\mathcal{D}}(u,u,v)\qquad \forall v\in \mathrm{H}_{\mathcal{D}}(\Omega)^d \tag{57}$$

the expression of which can easily be seen to read:

$$\widetilde{f}_K=\frac{1}{m_K}\left[\int_K f(x)\ \mathrm{d}x-\frac{\rho}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma\left(\frac{d_{L,\sigma}u_K+d_{K,\sigma}u_L}{d_\sigma}\cdot n_\sigma\right)\ u_L\right] \tag{58}$$

Taking $v=\widetilde{f}$ in (57), using the *a priori* bound (54), inequality (52) and the fact that, as $\mathrm{H}_{\mathcal{D}}(\Omega)$ is a finite dimensional space, the norms $\|\cdot\|_{\mathcal{D}}$ and $\|\cdot\|_{\mathrm{L}^2(\Omega)}$ are equivalent over $\mathrm{H}_{\mathcal{D}}(\Omega)$ (with an equivalence ratio depending on the mesh), one obtains that there exists $c_{\tilde{f},\mathcal{D}}$ only depending on d, Ω, ν, f and $\mathcal{D}$ and not on $\rho\in[0,1]$, such that the following inequality holds:

$$\|\widetilde{f}\|_{\mathrm{L}^2(\Omega)}\le c_{\tilde{f},\mathcal{D}}$$

Note that this bound may also be derived directly from the expression (58) by a discrete Cauchy-Schwarz inequality. It is clear, however, that there is no bound for $c_{\tilde{f},\mathcal{D}}$ as $h_{\mathcal{D}}$ tends to zero.

It is then sufficient to remark that (u,p) is solution to (21) with $\widetilde{f}$ instead of f, to conclude (56), applying theorem 4.4. □

We are now in position to prove the existence of at least one solution to scheme (49).

Theorem 5.2 (Existence of a discrete solution)**.** *Under hypotheses* (2)-(4)*, let* $\mathcal{D}$ *be an admissible discretization of* Ω *in the sense of definition 3.1, and* $\lambda \in (0,+\infty)$ *and* $\alpha \leq 1$ *be given. Then there exists at least one* $(u,p) \in (\mathrm{H}_{\mathcal{D}}(\Omega))^d \times \mathrm{H}_{\mathcal{D}}(\Omega)$*, solution to* (49)*.*

Proof. Let us define $V = \{(u,p) \in (\mathrm{H}_{\mathcal{D}}(\Omega))^d \times \mathrm{H}_{\mathcal{D}}(\Omega) \text{ s.t. } \int_\Omega p(x)\mathrm{d}x = 0\}$. Consider the continuous mapping $F \ : \ V \times [0,1] \to V$ such that, for a given $(u,p) \in V$ and $\rho \in [0,1]$, $(\hat{u},\hat{p}) = F(u,p,\rho)$ is defined by:

$$\left|\begin{array}{ll} \displaystyle\int_\Omega \hat{u}(x)\cdot v(x)\,\mathrm{d}x = \eta\int_\Omega u(x)\cdot v(x)\,\mathrm{d}x + \nu[u,v]_{\mathcal{D}} - \int_\Omega p(x)\ \mathrm{div}_{\mathcal{D}}(v)(x)\,\mathrm{d}x & \\ \qquad\qquad\qquad +\rho\ b_{\mathcal{D}}(u,u,v) - \displaystyle\int_\Omega f(x)\cdot v(x)\,\mathrm{d}x & \forall v \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \\ \displaystyle\int_\Omega \hat{p}(x)\cdot q(x)\,\mathrm{d}x = \int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)\ q(x)\,\mathrm{d}x + \langle p,q\rangle_{\mathcal{D},\lambda} & \forall q \in \mathrm{H}_{\mathcal{D}}(\Omega). \end{array}\right.$$

It is easily checked that the two above relations define a one to one function $F(.,.,.)$. Indeed, the value of $\hat{u}_K^{(i)}$ and $\hat{p}_K$ for a given $K \in \mathcal{M}$ and $i = 1,\ldots,d$ are readily obtained by setting $v^{(i)} = 1_K$, $v^{(j)} = 0$ for $j \neq i$, and $q = 1_K$.

The mapping $F(.,.,.)$ is continuous, and, for a given (u,p) such that $F(u,p,\rho) = (0,0)$, we can apply lemma 5.1, which proves that (u,p) is bounded independently on ρ. Since $F(u,p,0)$ is a bijective affine function of (u,p) (by corollary 4.6), the existence of at least one solution (u,p) to (49) follows by a topological degree argument (see [7] for the theory and [13, theorem 4.3] for a precise formulation of the abstract theorem used here). □

5.3. Convergence analysis

We first begin with two lemmas which are used in the proof of convergence of the scheme.

Lemma 5.3. *Under hypothesis* (2)*, let* $\theta > 0$ *be given and let* $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ *be a sequence of admissible discretizations of* Ω *in the sense of definition 3.1, such that* $\lim_{m\to\infty} h_{\mathcal{D}}^{(m)} = 0$ *and such that* $\mathrm{regul}(\mathcal{D}^{(m)}) \geq \theta$*, for all* $m \in \mathbb{N}$*. Let* $(u^{(m)})_{m\in\mathbb{N}}$ *and* $(v^{(m)})_{m\in\mathbb{N}}$ *be two sequences sastisfying the following assumptions:*

(1) $\forall m \in \mathbb{N}$, $u^{(m)} \in \mathrm{H}_{\mathcal{D}^{(m)}}(\Omega)^d$*, there exists* $\bar{u} \in \mathrm{H}_0^1(\Omega)^d$ *such that* $u^{(m)}$ *converges to* $\bar{u}$ *in* $(\mathrm{L}^2(\Omega))^d$ *as* m *tends to* ∞*,* $\|u^{(m)}\|_{\mathcal{D}}$ *remains bounded and:*

$$\lim_{m\to\infty}[u^{(m)},u^{(m)}]_{\mathcal{D}} = \int_\Omega \nabla\bar{u}(x) : \nabla\bar{u}(x)\ \mathrm{d}x \tag{59}$$

(2) $\forall m \in \mathbb{N}$, $v^{(m)} \in \mathrm{H}_{\mathcal{D}^{(m)}}(\Omega)^d$*, there exists* $\bar{v} \in \mathrm{H}_0^1(\Omega)^d$ *such that* $v^{(m)}$ *converges to* $\bar{v}$ *in* $(\mathrm{L}^2(\Omega))^d$ *as* m *tends to* ∞*, and* $\|v^{(m)}\|_{\mathcal{D}}$ *remains bounded.*

Then we have:

$$\lim_{m\to\infty}[u^{(m)},v^{(m)}]_{\mathcal{D}} = \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\ \mathrm{d}x \tag{60}$$

Proof. Let $\varphi \in (C_c^\infty(\Omega))^d$ be given. Thanks to assumptions on the sequence $(u^{(m)})_{m\in\mathbb{N}}$, the following classical result holds:

$$\lim_{m\to\infty}[u^{(m)}, P_{\mathcal{D}^{(m)}}\varphi]_{\mathcal{D}} = \int_\Omega \nabla\bar{u}(x) : \nabla\varphi(x)\ \mathrm{d}x \tag{61}$$

For the sake of completeness, let us recall the proof of this result with the tools introduced in Section 4.3. By a standard reordering of the summations:

$$\begin{aligned}
[u^{(m)}, P_{\mathcal{D}^{(m)}}\varphi]_{\mathcal{D}} &= \sum_{K\in\mathcal{M}} u_K^{(m)} \cdot \left[\sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\cap\mathcal{E}(K),\\ \sigma=K|L}} \frac{m_\sigma}{d_\sigma}\left[(P_{\mathcal{D}^{(m)}}\varphi)_K - (P_{\mathcal{D}^{(m)}}\varphi)_L\right] + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}(K)} \frac{m_\sigma}{d_{K,\sigma}}(P_{\mathcal{D}^{(m)}}\varphi)_K\right] \\
&= \sum_{K\in\mathcal{M}} u_K^{(m)} \cdot \int_K -\Delta\varphi(x)\ \mathrm{d}x + \sum_{K\in\mathcal{M}} u_K^{(m)} \cdot (R_\Delta(\varphi))_K \\
&= \int_\Omega u^{(m)} \cdot \Delta\varphi(x)\ \mathrm{d}x + \int_\Omega u^{(m)} \cdot R_\Delta(\varphi)(x)\ \mathrm{d}x
\end{aligned}$$

On one hand, because $u^{(m)}$ tends to $\bar{u}$ in $\mathrm{L}^2(\Omega)$, the first term tends to $\int_\Omega \bar{u}\cdot\Delta\varphi(x)\ \mathrm{d}x$; on the second hand, because $\|u^{(m)}\|_{\mathcal{D}}$ remains bounded as $m\to\infty$ and $\|R_\Delta(\varphi)\|_{-1,\mathcal{D}}$ tends to zero by theorem 4.7, the second one tends to zero, which proves (61).

Then developping $[u^{(m)} - P_{\mathcal{D}^{(m)}}\varphi, u^{(m)} - P_{\mathcal{D}^{(m)}}\varphi]_{\mathcal{D}^{(m)}}$ and using (59) yields:

$$\lim_{m\to\infty} \|u^{(m)} - P_{\mathcal{D}^{(m)}}\varphi\|^2_{\mathcal{D}^{(m)}} = \|\nabla(\bar{u}-\varphi)\|^2_{\mathrm{L}^2(\Omega)^d} \tag{62}$$

Let the sequence $(v^{(m)})_{m\in\mathbb{N}}$ satisfy the assumptions of the lemma; we denote by c_{v} a constant such that $\|v^{(m)}\|_{\mathcal{D}^{(m)}} \le c_{\mathrm{v}}$. We then have:

$$\begin{aligned}
&[u^{(m)}, v^{(m)}]_{\mathcal{D}^{(m)}} - \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\ \mathrm{d}x \\
&\qquad = [u^{(m)} - P_{\mathcal{D}^{(m)}}\varphi, v^{(m)}]_{\mathcal{D}^{(m)}} + [P_{\mathcal{D}^{(m)}}\varphi, v^{(m)}]_{\mathcal{D}^{(m)}} - \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\ \mathrm{d}x
\end{aligned}$$

The first term of the right hand side of the above equation is bounded by $c_{\mathrm{v}}\|u^{(m)} - P_{\mathcal{D}^{(m)}}\varphi\|_{\mathcal{D}^{(m)}}$ and the second one converges to $\int_\Omega \nabla\varphi(x) : \nabla\bar{v}(x)\mathrm{d}x$. Hence, thanks to (62), passing to the limit yields:

$$\limsup_{m\to\infty}\left|[u^{(m)}, v^{(m)}]_{\mathcal{D}^{(m)}} - \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\ \mathrm{d}x\right| \le \left(c_{\mathrm{v}} + \|\nabla\bar{v}\|_{\mathrm{L}^2(\Omega)^d}\right)\|\nabla(\bar{u}-\varphi)\|_{\mathrm{L}^2(\Omega)^d}$$

Since the above inequality holds for all $\varphi \in (C_c^\infty(\Omega))^d$, we let $\varphi \to \bar{u}$ in $\mathrm{H}_0^1(\Omega)^d$ to get:

$$\limsup_{m\to\infty}\left|[u^{(m)}, v^{(m)}]_{\mathcal{D}^{(m)}} - \int_\Omega \nabla\bar{u}(x) : \nabla\bar{v}(x)\mathrm{d}x\right| = 0$$

which concludes the proof. □

Lemma 5.4. *Under hypothesis* (2)*, let* $\theta > 0$ *be given and let* $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ *be a sequence of admissible discretizations of* Ω *in the sense of definition 3.1, such that* $\lim_{m\to\infty} h_{\mathcal{D}}^{(m)} = 0$ *and such that* $\mathrm{regul}(\mathcal{D}^{(m)}) \ge \theta$*, for all* $m\in\mathbb{N}$*. Let* $(u^{(m)})_{m\in\mathbb{N}}$ *be a sequence sastisfying the following assumptions:* $\forall m\in\mathbb{N}$, $u^{(m)} \in \mathrm{H}_{\mathcal{D}^{(m)}}(\Omega)$*, there exists* $\bar{u}\in\mathrm{H}_0^1(\Omega)$ *such that* $u^{(m)}$ *converges to* $\bar{u}$ *in* $\mathrm{L}^2(\Omega)$ *as* m *tends to* ∞ *and* $\|u^{(m)}\|_{\mathcal{D}}$ *remains bounded.*

(1) *For each edge σ of the mesh and $m \in \mathbb{N}$, we define:*

$$\begin{aligned} &\textit{If } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L \quad (\bar\nabla_{\mathcal{D}} u^{(m)})_\sigma = d\ \frac{1}{d_\sigma}\ [u_L^{(m)} - u_K^{(m)}]\ n_\sigma \\ &\textit{If } \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K) \quad (\bar\nabla_{\mathcal{D}} u^{(m)})_\sigma = d\ \frac{1}{d_\sigma}\ [0 - u_K^{(m)}]\ n_\sigma \end{aligned}$$

and we denote by $\bar\nabla_{\mathcal{D}} u^{(m)}$ the piecewise constant function equal to $(\bar\nabla_{\mathcal{D}} u^{(m)})_\sigma$ on the diamond cell D_σ associated to σ.
Then the sequence $\bar\nabla_{\mathcal{D}} u^{(m)}$ weakly converges to $\nabla \bar u$ in $\mathrm{L}^2(\Omega)$.

(2) *For each edge σ of the mesh and $m \in \mathbb{N}$, we define:*

$$\begin{aligned} &\textit{If } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L \quad u_\sigma^{(m)} = \frac{d_{L,\sigma}}{d_\sigma} u_K^{(m)} + \frac{d_{K,\sigma}}{d_\sigma} u_L^{(m)} \\ &\textit{If } \sigma \in \mathcal{E}_{\rm ext}, \quad u_\sigma^{(m)} = 0 \end{aligned}$$

and we denote by $\widetilde u^{(m)}$ the piecewise constant function equal to $u_\sigma^{(m)}$ on the diamond cell D_σ associated to σ.
Then the sequence $\widetilde u^{(m)}$ tends to $\bar u$ in $\mathrm{L}^p(\Omega)$, where $2 \le p < \infty$ if $d = 2$ and $2 \le p < 6$ if $d = 3$.

Proof. The proof of item (1) is given in [8, lemma 2]. For the proof of item (2), we first remark that, thanks to the discrete Sobolev inequalities $\|u\|_{L^r(\Omega)} \le c\|u\|_{\mathcal{D}^{(m)}}$ for $2 \le r \le \infty$ if $d = 2$ and for $2 \le r \le 6$ if $d = 3$ (see [6] or [9, p. 790]). We thus have:

$$\begin{aligned} \|\widetilde u^{(m)}\|^r_{\mathrm{L}^r(\Omega)} &= \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} m_{D_\sigma} \left| \frac{d_{L,\sigma}}{d_\sigma} u_K^{(m)} + \frac{d_{K,\sigma}}{d_\sigma} u_L^{(m)} \right|^r \le 2^{r-1} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} m_{D_\sigma} \left[|u_K^{(m)}|^r + |u_L^{(m)}|^r \right] \\ &\le 2^r \sum_{K\in\mathcal{M}} \left[\sum_{\sigma\in\mathcal{E}(K)} m_{D_\sigma} \right] |u_K^{(m)}|^r \le c\ \|u^{(m)}\|^r_{\mathrm{L}^r(\Omega)} \end{aligned}$$

where $c \in \mathbb{R}_+$ only depends on the regularity of the mesh, and m_{D_σ} denotes the measure of the subset D_σ; the sequence $(\widetilde u^{(m)})_{m\in\mathbb{N}}$ is therefore bounded in $\mathrm{L}^r(\Omega)$. On the other hand, we also have:

$$\|u^{(m)} - \widetilde u^{(m)}\|_{\mathrm{L}^2(\Omega)} \le \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} m_{D_\sigma} (u_K^{(m)} - u_L^{(m)})^2 \le c h_{\mathcal{D}^{(m)}}\ \|u^{(m)}\|_{\mathcal{D}^{(m)}}$$

and so the sequence $(\widetilde u^{(m)})_{m\in\mathbb{N}}$ tends to $\bar u$ in $\mathrm{L}^2(\Omega)$, which implies the result. □

We can now state the convergence result for the scheme (49).

Theorem 5.5 (Convergence of the scheme)**.** *Under hypotheses* (2)-(4)*, let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of Ω in the sense of definition 3.1, such that $h_{\mathcal{D}}^{(m)}$ tends to 0 as $m \to \infty$ and such that there exists $\theta > 0$ with* $\mathrm{regul}(\mathcal{D}^{(m)}) \ge \theta$*, for all $m \in \mathbb{N}$. Let $\lambda \in (0, +\infty)$ and $\alpha \le 1$ be given. Let, for all $m \in \mathbb{N}$, $(u^{(m)}, p^{(m)}) \in \mathrm{H}_{\mathcal{D}^{(m)}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}^{(m)}}(\Omega)$, be a solution to* (49) *with $\mathcal{D} = \mathcal{D}^{(m)}$. Then there exists a weak solution $\bar u$ of* (6) *and a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$, such that the corresponding subsequence of solutions $(u^{(m)})_{m\in\mathbb{N}}$ converges to $\bar u$ in $\mathrm{L}^2(\Omega)^d$.*

Proof. Thanks to the fact that the sequence $\|u^{(m)}\|_{\mathcal{D}^{(m)}}$ is bounded independently of m (estimate (54)), we obtain the existence of a subsequence of $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ and of $\bar u \in \mathrm{H}^1_0(\Omega)^d$ such that $(u^{(m)})_{m\in\mathbb{N}}$ converges to $\bar u$ in $\mathrm{L}^2(\Omega)^d$. We then again denote $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ this subsequence. Note that, in contrast with the Stokes problem, because of the lack of $\mathrm{L}^2(\Omega)$ estimate for the pressure, we may no longer prove the weak convergence of the discrete pressure.

With exactly the same arguments as for the Stokes problem, we get $\mathrm{div}\bar u(x) = 0$ for a.e. x in Ω.

Let $\varphi \in (C_c^\infty(\Omega))^d$ be such that $\mathrm{div}\varphi(x) = 0,\ \forall x \in \Omega$; our aim is to prove that $\bar u$ satifies the identity (6):

$$\eta\int_\Omega \bar u(x)\cdot\varphi(x) + \nu\int_\Omega \nabla\bar u(x) : \nabla\varphi(x)\,\mathrm{d}x\ + b(\bar u,\bar u,\varphi) = \int_\Omega f(x)\cdot\varphi(x)\,\mathrm{d}x$$

To this purpose, we need to write the analogue of this relation for each discretization $\mathcal{D}^{(m)}$, for a projection of φ onto the space $\mathrm{H}_{\mathcal{D}}(\Omega)^d$. Because we have only a very weak control on the pressure (from estimate (55)), this projection must be carefully designed to obtain that the discrete gradient term tends to zero. A suitable candidate to this purpose is obtained by making use of the discrete Stokes problem; we thus define, for all $m \in \mathbb{N}$, the pair $(u_\varphi^{(m)}, p_\varphi^{(m)})$ as the solution of (21), with the right hand side f equal to $-\nu\Delta\varphi + \eta\varphi$. The solution of the corresponding continuous problem is φ for the velocity and 0 for the pressure, so the conclusions of theorem 4.8 read:

(1) the sequence $(u_\varphi^{(m)})_{m\in\mathbb{N}}$ converges to φ in $\mathrm{L}^2(\Omega)^d$,
(2) the sequence $(\|u_\varphi^{(m)}\|_{\mathcal{D}^{(m)}})_{m\in\mathbb{N}}$ is bounded and:

$$\lim_{m\to\infty}[u_\varphi^{(m)}, u_\varphi^{(m)}]_{\mathcal{D}^{(m)}} = \int_\Omega \nabla\varphi : \nabla\varphi\ \mathrm{d}x$$

(3) the following convergence result holds:

$$\lim_{m\to\infty}|p_\varphi^{(m)}|_{\mathcal{D}^{(m)},\lambda} = 0 \tag{63}$$

We then introduce $u_\varphi^{(m)}$ as a test function in (49) with $\mathcal{D} = \mathcal{D}^{(m)}$:

$$\eta\int_\Omega u^{(m)}(x)\cdot u_\varphi^{(m)}(x)\ \mathrm{d}x + \nu[u^{(m)}, u_\varphi^{(m)}]_{\mathcal{D}^{(m)}} - \int_\Omega p^{(m)}(x)\ \mathrm{div}_{\mathcal{D}^{(m)}}(u_\varphi^{(m)})(x)\ \mathrm{d}x$$
$$+b_{\mathcal{D}^{(m)}}(u^{(m)}, u^{(m)}, u_\varphi^{(m)}) = \int_\Omega f(x)\cdot u_\varphi^{(m)}(x)\ \mathrm{d}x$$

Thanks to the definition of $u_\varphi^{(m)}$, we get that:

$$-\int_\Omega p^{(m)}(x)\ \mathrm{div}_{\mathcal{D}^{(m)}}(u_\varphi^{(m)})(x)\ \mathrm{d}x = \langle p^{(m)}, p_\varphi^{(m)}\rangle_{\mathcal{D}^{(m)},\lambda} \le |p^{(m)}|_{\mathcal{D}^{(m)},\lambda}\ |p_\varphi^{(m)}|_{\mathcal{D}^{(m)},\lambda},$$

and so, by (55) and (63), this term tends to zero. By lemma 5.3, we get that:

$$\lim_{m\to\infty}[u^{(m)}, u_\varphi^{(m)}]_{\mathcal{D}^{(m)}} = \int_\Omega \nabla\bar u(x) : \nabla\varphi(x)\ \mathrm{d}x$$

We also have, because of the convergence in $\mathrm{L}^2(\Omega)^d$ of both sequences $(u^{(m)})_{m\in\mathbb{N}}$ and $(u_\varphi^{(m)})_{m\in\mathbb{N}}$:

$$\lim_{m\to\infty}\int_\Omega u^{(m)}(x)\cdot u_\varphi^{(m)}(x)\ \mathrm{d}x = \int_\Omega \bar u(x)\cdot\varphi(x)\ \mathrm{d}x$$
$$\lim_{m\to\infty}\int_\Omega f(x)\cdot u_\varphi^{(m)}(x)\ \mathrm{d}x = \int_\Omega \bar f(x)\cdot\varphi(x)\ \mathrm{d}x$$

To conclude the proof, it is now sufficient to prove that:

$$\lim_{m\to\infty} b_{\mathcal{D}^{(m)}}(u^{(m)}, u^{(m)}, u_\varphi^{(m)}) = b(\bar u, \bar u, \varphi)$$

We remark that:

$$u_L \cdot v_K - u_K \cdot v_L = (u_L - u_K)\cdot \widetilde{v}_{KL} + (v_K - v_L)\cdot \widetilde{u}_{KL} \qquad \text{with:} \quad \left| \begin{array}{l} \widetilde{u}_{KL} = \dfrac{d_{L,\sigma} u_K + d_{K,\sigma} u_L}{d_{K|L}} \\ \widetilde{v}_{KL} = \dfrac{d_{L,\sigma} v_K + d_{K,\sigma} v_L}{d_{K|L}} \end{array} \right.$$

Hence $b_{\mathcal{D}}(u^{(m)}, u^{(m)}, u_\varphi^{(m)})$ reads:

$$\begin{aligned} b_{\mathcal{D}}(u^{(m)}, u^{(m)}, u_\varphi^{(m)}) &= \frac{1}{2}\sum_{i=1}^d \sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\cap\mathcal{E}(K),\\ \sigma=K|L}} m_\sigma \left[\widetilde{u}_\sigma^{(m)}\cdot n_\sigma\right] \left[u_L^{(m,i)}\ (u_\varphi^{(m,i)})_K\right] \\ &= \frac{1}{2}\sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma \left[\widetilde{u}_\sigma^{(m)}\cdot n_\sigma\right] \left[u_L^{(m,i)}\ (u_\varphi^{(m,i)})_K - u_K^{(m,i)}\ (u_\varphi^{(m,i)})_L\right] \\ &= \frac{1}{2}\sum_{i=1}^d \underbrace{\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma \left[\widetilde{u}_\sigma^{(m)}\cdot n_\sigma\right] \left[u_L^{(m,i)} - u_K^{(m,i)}\right] (\widetilde{u}_\varphi^{(m,i)})_\sigma}_{T_1^{(i)}} \\ &\qquad + \frac{1}{2}\sum_{i=1}^d \underbrace{\sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_\sigma \left[\widetilde{u}_\sigma^{(m)}\cdot n_\sigma\right] \left[(u_\varphi^{(m,i)})_K - (u_\varphi^{(m,i)})_L\right] \widetilde{u}_\sigma^{(m,i)}}_{T_2^{(i)}} \end{aligned}$$

The term $T_1^{(i)}$ equivalenty reads:

$$T_1^{(i)} = \sum_{\sigma\in\mathcal{E}_{\rm int},\ \sigma=K|L} m_{D_\sigma}\ (u_\varphi^{(m,i)})_\sigma\ \widetilde{u}_\sigma^{(m)} \cdot \left[d\ \frac{u_L^{(m,i)} - u_K^{(m,i)}}{d_\sigma} n_\sigma\right] = \int_\Omega \widetilde{u}_\varphi^{(m,i)}(x)\ \widetilde{u}^{(m)}(x)\cdot \bar\nabla_{\mathcal{D}^{(m)}} u^{(m,i)}\ {\rm d}x$$

and so, by lemma 5.4, we can pass to the limit in the above equation to obtain:

$$\lim_{m\to\infty} T_1^{(i)} = \int_\Omega \varphi^{(i)} \bar u \cdot \nabla u\ {\rm d}x$$

By the same arguments, we get:

$$\lim_{m\to\infty} T_2^{(i)} = -\int_\Omega \bar u^{(i)} \bar u \cdot \nabla\varphi\ {\rm d}x$$

Since ${\rm div}\bar u = 0$, both limits are equal and the proof of convergence is complete. □

6. Numerical tests

The aim of this section is to check the validity of the theoretical analysis against a practical test case for which an analytical solution can be exhibited. This solution is built as follows. We choose a streamfunction and a geometrical domain such that homogeneous Dirichlet conditions hold:

$$\varphi = 1000 \ [\ x\,(1-x)\,y\,(1-y) \ ]^2 \quad , \qquad \Omega =]0,1[\times]0,1[\quad , \qquad \bar{u} = \begin{bmatrix} \dfrac{\partial \varphi}{\partial y} \\ -\dfrac{\partial \varphi}{\partial x} \end{bmatrix}$$

we pick an arbitrary pressure in $\mathrm{L}^2(\Omega)$:

$$\bar{p} = 100\ (x^2 + y^2 - \frac{2}{3})$$

and the right hand side f is computed so $\bar{u}$ and $\bar{p}$ are solutions to the stationary Navier-Stokes equations, written in dimensional form:

$$\rho(\bar{u}\nabla)\bar{u} - \mu\Delta\bar{u} + \nabla\bar{p} = f$$

To obtain the numerical results displayed here, the practical implementation has been performed using the software object-oriented component library PELICANS, developed at IRSN [22].

The velocity and pressure errors are defined respectively as:

$$e_K^{(i)} = u_K^{(i)} - \bar{u}^{(i)}(x_K) \quad , \qquad \epsilon_K = p_K - \bar{p}(x_K)$$

This pressure error definition is not the same as in the analysis; however it is easy to see from theorem 4.7 that, for a regular pressure field (for instance, in $\mathrm{H}^2(\Omega)$), this definition equivalently leads to a first order convergence.

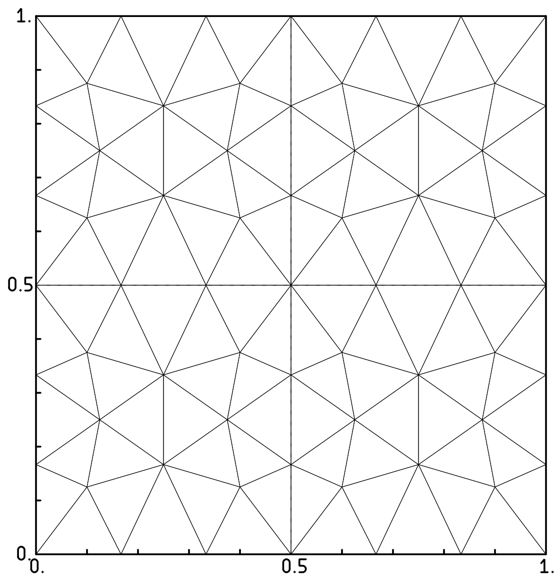


FIGURE 3. Coarsest mesh of the computational domain in clusters

The partition of the computational domain in clusters is built by first splitting the domain into sub-squares and then cutting each sub-square into 26 triangles, all having angles of at most 80° (corresponding to figure 5 – bbbb in [2]). The coarsest one is displayed on figure 3. Control volumes are then obtained by cutting each cluster into four similar triangles, the vertices of which are located on the mid-points of each edge, as shown in figure 4.

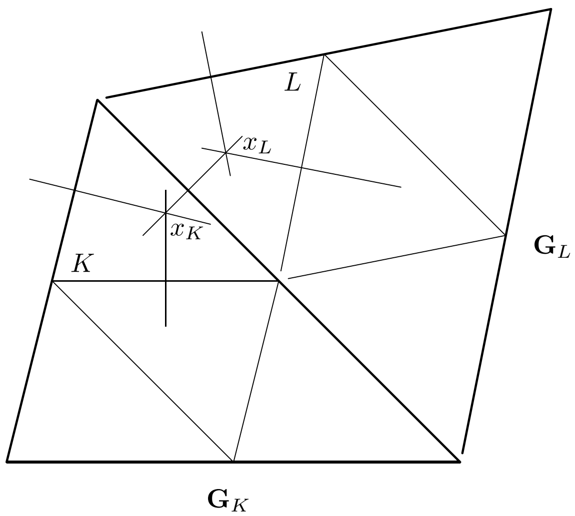


FIGURE 4. Exemple of clustered simplicial mesh, as used in numerical experiments.

We first begin by the Stokes problem, obtained by setting the density ρ to zero and the viscosity μ to 1. The norms of the errors between the numerical and the exact solution are displayed on figure 5; we observe a first order convergence for the velocity and the pressure in respectively discrete H^1 and L^2 norms, and a second order convergence for the velocity in the discrete L^2 norm.

We then turn to Navier-Stokes equations, setting $\rho = 100$ and $\mu = 1$ which, in view of the value of the velocity, leads to a Reynolds number Re in the range of Re$\approx$ 1000. We observe a noticeable loss of accuracy for the pressure, the convergence of which is, as a counterpart, faster than 1 at high value of the mesh size $h_{\mathcal{D}}$. For coarse grids, pressure oscillations are observed, which does not affect the velocity field; these pressure oscillations does not appear for the Stokes problem, neither when using a mesh based on rectangles (to construct such meshes, clusters are built first by a structured gridding of the domain, then control volumes are obtained by cutting each (rectangular) cluster in four, along the lines joining the mid-edge points). Once again, a first and second order convergence is obtained for the velocity in respectively discrete H^1 and L^2 norms.

Finally, numerical experiments show that the accuracy of the results is almost insensitive to the stabilization parameter λ, as soon as $\lambda \geq 0.1$: indeed, for the studied Navier-Stokes case and an intermediate mesh ($h_{\mathcal{D}} \approx 0.02$), only a difference of less than 20% on the magnitude of the errors is obtained when varying λ up to 10. For lower values of λ, the accuracy of the pressure is degradated first, then the velocity is affected; for $\lambda = 0.001$, the error is multiplied by 2 for the velocity and by 7 for the pressure.

7. CONCLUSION

We have presented and analysed in this paper a novel cell-centered colocated finite volume scheme for incompressible flow problems. This scheme is shown to be stable and convergent for the Navier-Stokes equations; moreover, we prove that it is first-order accurate in natural energy norms for the Stokes problem. Numerical experiments confirm the analysis and show, in addition, that the scheme is still first order accurate for a high Reynolds number problem; in addition, a second order convergence for the velocity in a discrete L^2 norm is

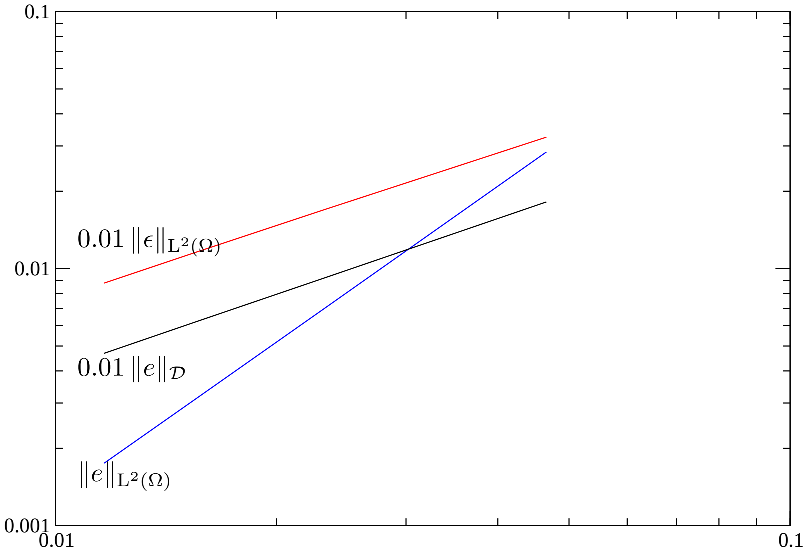


FIGURE 5. Errors for the velocity and the pressure obtained for the Stokes problem, as a function of the mesh parameter $h_{\mathcal{D}}$.

observed in any case. Unfortunately, these properties only hold for a particular class of meshes, the so–called super–admissible meshes, see definition 3.1, which is rather restrictive in practice; getting rid of these limitations is clearly a topic of interest for future work.

One underlying argument of this analysis that the pair of discrete spaces associating the classical cell-centered approximation for the velocities and cluster-wide constant pressures is *inf-sup* stable; to our knowledge, this is the first result of this type for finite volume discretizations.

The present work is already extended in practical applications to unsteady problems, also involving heat transfer, either within the framework of the Boussinesq approximation or using the more general asymptotic model for low Mach-number flows. These problems should deserve more attention in the future, both from a theoretical point of view as for the design of efficient numerical solvers.

Appendix A. Proof of the consistency results

We begin this section by stating a trace lemma which will be used in the following developments.

Lemma A.1 (A trace inequality)**.** *Assume that $d = 3$, and let $\mathcal{M}$ be an admissible mesh in the sense of Definition 3.1. Let σ be a given edge of the mesh and K be a control volume the boundary of which contains σ. We denote by $D_{K,\sigma,1/2}$ the volume defined by:*

$$D_{K,\sigma,1/2} = \{tx + (1-t)x_K,\ x \in \sigma,\ t \in (\frac{1}{2}, 1)\}$$

Let v be a function of $\mathrm{H}^1(D_{K,\sigma,1/2})$. Then there exists a constant $c_{\mathrm{tr}} \leq \sqrt{10}$ such that the following bound holds:

$$\|v\|_{\mathrm{L}^2(\sigma)} \leq c_{\mathrm{tr}}\ \frac{1}{d_{K,\sigma}^{1/2}}\ \Big[\|v\|_{\mathrm{L}^2(D_{K,\sigma,1/2})} + h_\sigma |\nabla v|_{\mathrm{L}^2(D_{K,\sigma,1/2})}\Big] \tag{64}$$

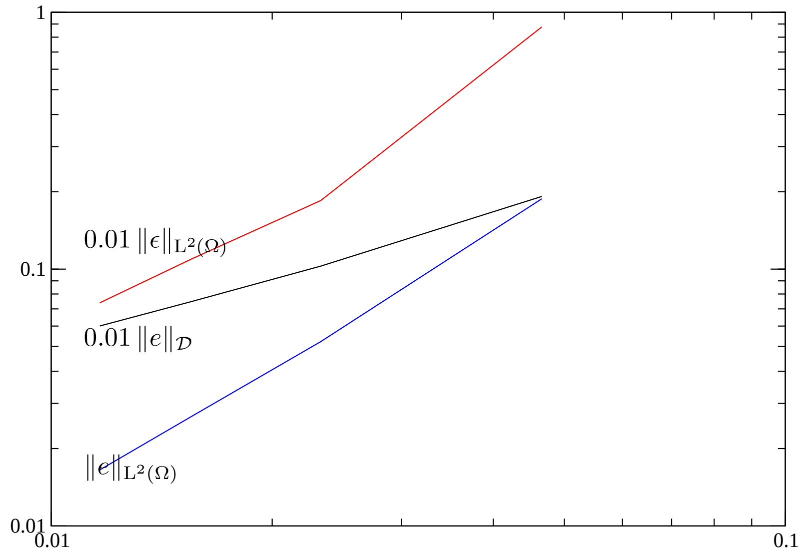


FIGURE 6. Errors for the velocity and the pressure obtained in the Navier-Stokes case (Re ≈1000), as a function of the mesh parameter $h_{\mathcal{D}}$.

where $h_\sigma = \mathrm{diam}(\sigma) + d_{K,\sigma}$. *Note, in particular, that* $h_\sigma \leq 2\, h_K$.

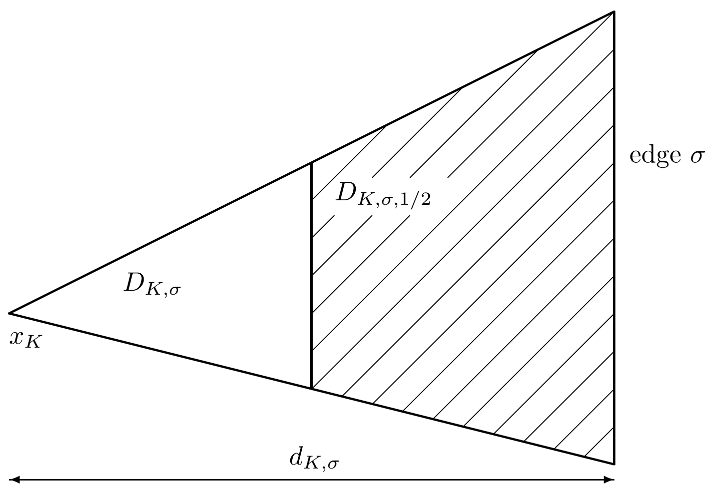


FIGURE 7. Sketch of the geometrical configuration used for lemma A.1

Proof. Let an edge of the mesh σ be given, K be an adjacent element and v be a function of $\mathrm{H}^1(D_{K,\sigma,1/2})$. Without loss of generality, we suppose that σ is a part of the plane $x^{(1)} = 0$ and that x_K is located at

$(-d_{K,\sigma},0,0)^t$ (see figure 7). Let us define the following mapping:

$$F \;:\; \left| \begin{array}{lcl} [0,1]\times\sigma & \rightarrow & D_{K,\sigma} \\ (t,y) & \mapsto & x=(t-1)\,x_K+t\,y \end{array} \right.$$

This mapping is regular and we have: $\mathrm{d}x = t^2 d_{K,\sigma}\,\mathrm{d}t\,\mathrm{d}\gamma(y)$. In addition, the following elementary geometrical relation holds:

$$t=\frac{x^{(1)}+d_{K,\sigma}}{d_{K,\sigma}}, \qquad \text{and} \qquad \mathrm{d}t\,\mathrm{d}\gamma(y)=\frac{d_{K,\sigma}}{(x^{(1)}+d_{K,\sigma})^2}\,\mathrm{d}x$$

For any y on σ, we have:

$$\begin{aligned} v(y)^2 &= \int_{1/2}^1 \frac{\partial}{\partial t}\left[(2t-1)v^2(F(t,y))\right]\,\mathrm{d}t \\ &= 2\int_{1/2}^1 v^2(F(t,y))\,\mathrm{d}t + 2\int_{1/2}^1 (2t-1)\ \left[\nabla v(F(t,y))\cdot(y-x_K)\right]\ v(F(t,y))\,\mathrm{d}t \end{aligned}$$

Integrating on σ, we thus get:

$$\begin{aligned} \int_\sigma v(y)^2\,\mathrm{d}\gamma(y) = &\underbrace{2\int_\sigma\int_{1/2}^1 v^2(F(t,y))\,\mathrm{d}t\,\mathrm{d}\gamma(y)}_{T_1} \\ &+\underbrace{2\int_\sigma\int_{1/2}^1 (2t-1)\ \left[\nabla v(F(t,y))\cdot(y-x_K)\right]\ v(F(t,y))\,\mathrm{d}t\,\mathrm{d}\gamma(y)}_{T_2} \end{aligned}$$

On one hand, the first term of this relation can be estimated as follows:

$$T_1 = 2\int_{D_{K,\sigma,1/2}} v(x)^2\ \frac{d_{K,\sigma}}{(x^{(1)}+d_{K,\sigma})^2}\,\mathrm{d}x$$

As $\forall x\in D_{K,\sigma,1/2},\ x^{(1)}\geq -d_{K,\sigma}/2$, the following inequality holds:

$$\forall x\in D_{K,\sigma,1/2}, \qquad \frac{d_{K,\sigma}}{(x^{(1)}+d_{K,\sigma})^2}\leq\frac{4}{d_{K,\sigma}}$$

and thus:

$$|T_1|\leq\frac{8}{d_{K,\sigma}}\|v\|^2_{\mathrm{L}^2(D_{K,\sigma,1/2})}$$

On the second hand, as $\forall x\in D_{K,\sigma,1/2},\ \|x-x_K\|\leq \mathrm{diam}(\sigma)+d_{K,\sigma}=h_\sigma$, the term T_2 can be bounded by:

$$\begin{aligned} T_2 &\leq 2\,h_\sigma\int_\sigma\int_{1/2}^1 \|\nabla v(F(t,y))\|\ |v(F(t,y))|\,\mathrm{d}t\,\mathrm{d}\gamma(y) \\ &= 2\,h_\sigma\int_{D_{K,\sigma,1/2}} \|\nabla v(x)\|\ |v(x)|\ \frac{d_{K,\sigma}}{(x^{(1)}+d_{K,\sigma})^2}\,\mathrm{d}x \\ &\leq \frac{8}{d_{K,\sigma}}\ h_\sigma\ \|\nabla v\|_{\mathrm{L}^2(D_{K,\sigma,1/2})}\ \|v\|_{\mathrm{L}^2(D_{K,\sigma,1/2})} \end{aligned}$$

Collecting the bounds of T_1 and T_2, we get:

$$\|v\|^2_{\mathrm{L}^2(\sigma)} \leq \frac{8}{d_{K,\sigma}} \left[\|v\|^2_{\mathrm{L}^2(D_{K,\sigma,1/2})} + h_\sigma \; \|\nabla v\|_{\mathrm{L}^2(D_{K,\sigma,1/2})} \; \|v\|_{\mathrm{L}^2(D_{K,\sigma,1/2})} \right]$$

and thus, thanks to Young's inequality, we obtain $\forall \alpha > 0$:

$$\|v\|^2_{\mathrm{L}^2(\sigma)} \leq \frac{8}{d_{K,\sigma}} \left[(1 + \frac{\alpha}{2}) \|v\|^2_{\mathrm{L}^2(D_{K,\sigma,1/2})} + h_\sigma^2 \, \frac{1}{2\alpha} \; \|\nabla v\|^2_{\mathrm{L}^2(D_{K,\sigma,1/2})} \right]$$

Choosing $\alpha = 1/2$ yields the result. □

In two dimensions, the following similar estimate is proven in [25]:

$$\|v\|_{\mathrm{L}^2(\sigma)} \leq \sqrt{2} \, \left(\frac{m_\sigma}{m_{D_{K,\sigma}}} \right)^{1/2} \; \left[\|v\|_{\mathrm{L}^2(D_{K,\sigma})} + h_\sigma |\nabla v|_{\mathrm{L}^2(D_{K,\sigma})} \right] \tag{65}$$

We recall the following Poincaré inequality, proved by Payne and Weinberger [21] and valid for any convex domain ω:

$$\forall \phi \in \mathrm{H}^1(\omega) \text{ such that } \int_\omega \phi(x) \; \mathrm{d}x = 0, \quad \|\phi\|_{\mathrm{L}^2(\omega)} \leq \frac{\mathrm{diam}(\omega)}{\pi} \; |\phi|_{\mathrm{H}^1(\omega)} \tag{66}$$

We are now in position to give a bound of elementary (*i.e.* related to a single edge or control volume) consistency residuals; this is the aim of the following two lemmas.

Lemma A.2. *We suppose that the assumption* (2) *holds. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and let $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Throughout the statement of this lemma, c stands for a positive real number only depending on d, Ω and θ.*

Let v be a function of $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$*. Then the following bounds hold:*

$$\forall \sigma \in (\mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}), \quad |R_{\Delta,\sigma}(v)| \leq c \; \frac{[\displaystyle\sum_{\sigma \in \mathcal{E}(K)} h_K]^2}{(m_\sigma \mathrm{d}_\sigma)^{1/2} \; \mathrm{d}_\sigma} \; |v|_{\mathrm{H}^2(D_\sigma)} \tag{67}$$

$$\forall K \in \mathcal{M}, \qquad |R_{\mathrm{o},K}(v)| \leq c \; h_K \; m_K^{-1/2} \; \|v\|_{\mathrm{H}^2(K)} \tag{68}$$

If the mesh is super–admissible in the sense of definition 3.1, we have:

$$\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \qquad |R_{\mathrm{div},\sigma}(v)| \leq c \; \frac{[\displaystyle\sum_{\sigma \in \mathcal{E}(K)} h_K]^2}{(m_\sigma \mathrm{d}_\sigma)^{1/2}} \; |v|_{\mathrm{H}^2(D_\sigma)} \tag{69}$$

Proof. By definition of $R_{\Delta,\sigma}(v)$, the bound (67) is equivalent to:

$$\left| \frac{1}{d_\sigma} \; [P_{\mathcal{D}}(v)_L - P_{\mathcal{D}}(v)_K] - \frac{1}{m\sigma} \int_\sigma \nabla v(x) \cdot n_\sigma \; \mathrm{d}\gamma(x) \right| \leq c \; \frac{[\displaystyle\sum_{\sigma \in \mathcal{E}(K)} h_K]^2}{(m_\sigma \mathrm{d}_\sigma)^{1/2} \; \mathrm{d}_\sigma} \; |v|_{\mathrm{H}^2(D_\sigma)} \tag{70}$$

(where $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\mathrm{int}}$ and $\sigma \in \mathcal{E}_K$ and $P_{\mathcal{D}}(v)_L = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}}$) which is proven in [9, pp. 786-789].

The bound (68) is equivalent to:

$$\left| P_{\mathcal{D}}(v)_K - \frac{1}{m_K} \int_K v(x) \; \mathrm{d}x \right| \leq c \; h_K \; m_K^{-1/2} \; \|v\|_{\mathrm{H}^2(K)} \tag{71}$$

A stronger version of the estimate (71) is given in [17, lemma 3.3, equation 3.26].

We now turn to the proof of (69). By definition of $R_{\mathrm{div},\sigma}$, this bound is equivalent to:

$$|\frac{d_{L,\sigma}}{d_\sigma}\ P_{\mathcal{D}}(v)_K + \frac{d_{K,\sigma}}{d_\sigma}\ P_{\mathcal{D}}(v)_L - \frac{1}{m_\sigma}\int_\sigma v(x)\ \mathrm{d}\gamma(x)| \leq c\ \frac{[h_K+h_L]^2}{(m_\sigma \mathrm{d}_\sigma)^{1/2}}\ |v|_{\mathrm{H}^2(D_\sigma)} \tag{72}$$

Let v be now a function of $\mathrm{C}^2(\bar{D}_\sigma)$. The two following Taylor expansions hold for any x of $\sigma = K|L$:

$$v(x_K) = v(x) + \nabla v(x)\cdot(x_K - x) + \int_0^1 [H(v)(tx+(1-t)x_K)\cdot(x_K-x)]\cdot(x_K-x)\,t\mathrm{d}t$$
$$v(x_L) = v(x) + \nabla v(x)\cdot(x_L - x) + \int_0^1 [H(v)(tx+(1-t)x_L)\cdot(x_L-x)]\cdot(x_L-x)\,t\mathrm{d}t$$

where $H(v)(x)$ is the Hessian matrix of v at point x.
Multiplying the first relation by $d_{\sigma,L}/d_\sigma$, the second one by $d_{K,\sigma}/d_\sigma$, summing and integrating over σ yields:

$$\frac{1}{m_\sigma}\int_\sigma v(x)\ \mathrm{d}\gamma(x) = \frac{d_{\sigma,L}}{d_\sigma}\, v(x_K) + \frac{d_{K,\sigma}}{d_\sigma}\, v(x_L) + \underbrace{\frac{1}{\sigma}\int_\sigma \nabla v(x)(x_G - x)\ \mathrm{d}\gamma(x)}_{T_1} + \frac{d_{\sigma,L}}{d_\sigma}\, R_{K,\sigma} + \frac{d_{K,\sigma}}{d_\sigma}\, R_{K,\sigma}$$

where $x_G = \dfrac{d_{\sigma,L}}{d_\sigma}\, x_K + \dfrac{d_{K,\sigma}}{d_\sigma}\, x_L$ is the mass center of σ, thanks to the fact that the mesh is assumed to be super–admissible, and:

$$R_{K,\sigma} = \frac{1}{\sigma}\int_\sigma\int_0^1 [H(v)(tx+(1-t)x_K)\cdot(x_K-x)]\cdot(x_K-x)\,t\mathrm{d}t\ \mathrm{d}\gamma(x),$$
$$R_{K,\sigma} = \frac{1}{\sigma}\int_\sigma\int_0^1 [H(v)(tx+(1-t)x_L)\cdot(x_L-x)]\cdot(x_L-x)\,t\mathrm{d}t\ \mathrm{d}\gamma(x)$$

The following bound of these quantities is given in [9, pp. 786-789]:

$$|R_{K,\sigma}| \leq c\frac{h_K^2}{(m_\sigma d_{K,\sigma})^{1/2}}\ |v|_{\mathrm{H}^2(D_{K,\sigma})}, \qquad |R_{L,\sigma}| \leq c\frac{h_L^2}{(m_\sigma d_{L,\sigma})^{1/2}}\ |v|_{\mathrm{H}^2(D_{L,\sigma})}$$

where c only depends on the space dimension d. On the other hand, we have:

$$T_1 = \frac{1}{\sigma}\int_\sigma \nabla v(x)(x_G - x)\ \mathrm{d}\gamma(x) = \frac{1}{\sigma}\int_\sigma \nabla(v(x)-p(x))(x_G-x)\ \mathrm{d}\gamma(x)$$

for any linear polynomial $p(\cdot)$. Using the Cauchy-Schwarz inequality then the convenient trace lemma (*i.e.* choosing either K or L and applying (65) for $d=2$ and (64) for $d=3$), we obtain:

$$\begin{aligned} |T_1| &\leq \frac{\max(h_K,h_L)}{m_\sigma^{1/2}}\|\nabla(v(x)-p(x))\|_{\mathrm{L}^2(\sigma)} \\ &\leq c\frac{\max(h_K,h_L)}{m_\sigma^{1/2}}\ \frac{1}{max(d_{K,\sigma},d_{L,\sigma})^{1/2}}\ \left[\|\nabla(v(x)-p(x))\|_{\mathrm{L}^2(D_\sigma)} + \max(h_K,h_L)|v|_{\mathrm{H}^2(D_\sigma)}\right] \end{aligned}$$

where c only depends on the regularity of the mesh. Choosing for p the function defined by:

$$p(x) = \sum_{i=1}^d x^{(i)}\ \frac{1}{m_{D_\sigma}}\int_{D_\sigma}\frac{\partial v}{\partial x^{(i)}}(x)\ \mathrm{d}x$$

and applying the Poincaré inequality (66) yields (73) and concludes the proof.

□

Lemma A.3. *We suppose that hypotheses* (2) *holds. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Throughout the statement of this lemma, c stands for a positive real number only depending on d, Ω and θ. Let $v \in \mathrm{H}^1(\Omega)$. Then, for each edge σ of the mesh and each control volume K such that $\sigma \subset \bar{K}$, we have:*

$$|\frac{1}{m_\sigma} \int_\sigma v(x)\ \mathrm{d}\gamma(x) - P_{\mathcal{M}}(v)_K| \leq c\ \left[\frac{h_K}{m_\sigma}\right]^{1/2}\ |v|_{\mathrm{H}^1(K)} \tag{73}$$

Consequently, for each pair of neighbouring control volumes K and L of the mesh, the following estimate holds:

$$|P_{\mathcal{M}}(v)_L - P_{\mathcal{M}}(v)_K| \leq c\ \left[\frac{h_K + h_L}{m_{K|L}}\right]^{1/2}\ |v|_{\mathrm{H}^1(K\cup L)} \tag{74}$$

Proof. The results (73)-(74) are proven for $d = 2$ in [9, pp. 777-779]. We provide here a proof of this latter estimate valid for $d = 2$ and $d = 3$. Let σ be an edge of the mesh, K one control volume such that $\sigma \subset \bar{K}$ and v a function of $\mathrm{H}^1(K)$. We have:

$$R = P_{\mathcal{M}}(v)_K - \frac{1}{m_\sigma} \int_K v(x)\, \mathrm{d}\sigma = \frac{1}{m_\sigma} \int_K (P_{\mathcal{M}}(v)_K - v(x))\, \mathrm{d}\sigma$$

By the Cauchy-Schwarz inequality then the estimates (65) (for d=2) or (64) (for d=3), as either $D_{K,\sigma}$ or $D_{K,\sigma,1/2}$ are included in K, we get:

$$|R| \leq \frac{1}{m_\sigma^{1/2}}\ \|v\|_{\mathrm{L}^2(\sigma)} \leq c\ \frac{1}{(m_\sigma d_{K,\sigma})^{1/2}}\ \left[\|v - P_{\mathcal{M}}(v)_K\|_{\mathrm{L}^2(K)} + h_K \|\nabla v\|_{\mathrm{L}^2(K)}\right]$$

and thus, by the Poincaré inequality (66):

$$|R| \leq c \frac{h_K}{(m_\sigma d_{K,\sigma})^{1/2}}\ \|\nabla v\|_{\mathrm{L}^2(K)}$$

and the estimate (73) follows using regularity assumptions for the mesh; the inequality (74) is then an easy consequence of this result and the triangular inequality.

□

Corollary A.4. *We suppose that hypotheses* (2) *holds. Let $\mathcal{D}$ be an admissible discretization of Ω in the sense of definition 3.1 and $\theta > 0$ be such that* $\mathrm{regul}(\mathcal{D}) > \theta$*. Throughout the statement of this lemma, c stands for a positive real number only depending on d, Ω and θ. Let $v \in \mathrm{H}^1(\Omega)$. Then the following bounds hold:*

$$\forall \sigma \in (\mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}), \quad |R_{\mathrm{grad},\mathcal{M},\sigma}| \leq c\ \frac{[\sum_{\sigma\in\mathcal{E}(K)} h_K]^{1/2}}{m_\sigma^{1/2}}\ |v|_{\mathrm{H}^1(\cup_{\sigma\in\mathcal{E}(K)} K)} \tag{75}$$

$$\forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad |R_{\mathrm{stab},\mathcal{M},\sigma}| \leq c\ \frac{[\sum_{\sigma\in\mathcal{E}(K)} h_K]^{\alpha+1/2}}{m_\sigma^{1/2}}\ |v|_{\mathrm{H}^1(\cup_{\sigma\in\mathcal{E}(K)} K)} \tag{76}$$

$$\forall \sigma \in (\mathcal{E}_{\mathrm{int}} \cup \mathcal{E}_{\mathrm{ext}}), \quad |R_{\mathrm{grad},\mathcal{G},\sigma}| \leq c\ \left[\max_{K\in(G_\sigma)} (h_K)\ \frac{1}{m_\sigma}\right]^{1/2}\ |v|_{\mathrm{H}^1(C_\sigma)} \tag{77}$$

where G_σ and C_σ are defined as follows: if $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, then $G_\sigma = G_K \cup G_L$ and $C_\sigma = C_{G_K} \cup C_{G_L}$ and, if $\sigma \in \mathcal{E}_{\rm ext}$, $\sigma \in \mathcal{E}(K)$, $G_\sigma = G_K$ and $C_\sigma = C_{G_K}$; note that, if σ is an internal edge of a cluster, $G_K = G_L$.

Proof. The estimate (75) is the same relation as (73) for an external edge or (74) for an internal one. The bound (76) follows directly from (74).

Let us prove the estimate (77). Let σ be an edge of the mesh and K a control volume such that $\sigma \in \mathcal{E}(K)$; we denote by $m_{C_{G_K}}$ the measure of C_{G_K}. We suppose first that σ is an external edge of the mesh. By definition, we have:

$$|R_{{\rm grad},\mathcal{G},\sigma}| = |\frac{1}{m_{C_{G_K}}} \int_{C_{G_K}} v(x)\ {\rm d}x - \frac{1}{m_\sigma} \int_\sigma v(x)\ {\rm d}\gamma(x)|$$

Decomposing the measure of C_{G_K} and the integral over C_{G_K}, we get:

$$\begin{aligned} |R_{{\rm grad},\mathcal{G},\sigma}| &= |\sum_{L\in G_K} \frac{m_L}{(\sum\limits_{L\in G_K} m_K)} \frac{1}{m_L} \int_L v(x)\ {\rm d}x - \frac{1}{m_\sigma}\int_\sigma v(x)\ {\rm d}\gamma(x)| \\ &= |\sum_{L\in G_K} \frac{m_L}{(\sum\limits_{L\in G_K} m_L)} \left[\underbrace{\frac{1}{m_L}\int_L v(x)\ {\rm d}x - \frac{1}{m_K}\int_K v(x)\ {\rm d}x}_{T_{K,L}} \right] \\ &\qquad + \underbrace{\frac{1}{m_K}\int_K v(x)\ {\rm d}x - \frac{1}{m_\sigma}\int_\sigma v(x)\ {\rm d}\gamma(x)\,|}_{T_2} \end{aligned}$$

The latter difference T_2 is bounded by (73), and, if K can be chosen such as, for each L in G_K, K and L are two neighbouring control volumes, each difference $T_{K,L}$ in the sum is bounded by (74); otherwise, this difference may be decomposed in a sum of differences of the mean value of $v(\cdot)$ over two neighbouring control volumes, the number of terms of this sum being bounded by the number of control volumes in the cluster G_K. This completes the proof of (77), in the case where $\sigma \in \mathcal{E}_{\rm ext}$. When σ is an internal edge of a cluster, the quantities $P_\mathcal{G}(v)_K$ and $P_\mathcal{G}(v)_L$ are identical, and the definition of $R_{{\rm grad},\mathcal{G},\sigma}$ is the same as in the previous case. Finally, when σ is at the boundary of two clusters, the bound (77) follows from the same argument, using the triangular inequality. □

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